

AN EXAMPLE OF RAPID EVOLUTION OF COMPLEX LIMIT CYCLES

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ABSTRACT. In the current article we study complex cycles of higher multiplicity in a specific polynomial family of holomorphic foliations in the complex plane. The family in question is a perturbation of an exact polynomial one-form giving rise to a foliation by Riemann surfaces. In this setting, a complex cycle is defined as a nontrivial element of the fundamental group of a leaf from the foliation. In addition to that, we introduce the notion of a multi-fold cycle and show that in our example there exists a limit cycle of any multiplicity. Furthermore, such a cycle gives rise to a one-parameter family of cycles continuously depending on the perturbation parameter. As the parameter decreases in absolute value, the cycles from the continuous family escape from a very large subdomain of the complex plane.

1. INTRODUCTION

Limit cycles of planar polynomial vector fields have long been a focus of extensive research. For instance, one of the major problems in this area of dynamical systems is the famous Hilbert's 16th problem [12] asking about the number and the location of the limit cycles of a polynomial vector field of degree n in the plane. Since the original Hilbert's problem continues to be very persistent, some simplifications have been considered as well. Among them is the so called infinitesimal Hilbert's 16 problem [12], [13] concerned with the number of limit cycles that can bifurcate from periodic solutions of a polynomial Hamiltonian planar system by a small polynomial perturbation. Recently, an answer to this question has been given in an article by Binyamini, Novikov and Yakovenko [2].

When studying a planar polynomial vector field, an extension to the complex domain proves to be helpful, an idea that can be attributed to Petrovskii and Landis [15], [16]. In this way a polynomial complex vector field is obtained and the holomorphic curves tangent to it form a partition of the complex plane by Riemann surfaces, called a *polynomial complex foliation with singularities*, or in short *polynomial complex foliation* [12], [13].

Following the idea of complexification, polynomial deformations of planar Hamiltonian vector fields could be extended to \mathbb{C}^2 . More precisely one could consider the complex line field

$$\ker(dH + \varepsilon\omega) \tag{1}$$

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with a one-form $\omega = Adx + Bdy$, where A, B and $H \in \mathbb{C}[x, y]$ are polynomials with complex coefficients and ε is a small complex parameter.

For the purposes of the current study, we focus our attention on a specific example. Let H be the simple polynomial

$$H = x^2 + y^2.$$

Choose polynomial one-forms ω_1 and ω_2 as follows:

$$\omega_1 = (H - 1)(ydx - xdy) \quad \text{and} \quad \omega_2 = y dH.$$

Consider the two parameter family of complex line fields

$$F_{a,\varepsilon} = \ker(dH + \varepsilon(\omega_1 + a\omega_2)), \quad (2)$$

where ε and a are the parameters. Notice that the family is of the form (1).

As mentioned earlier, the holomorphic curves tangent to $F_{a,\varepsilon}$ form a foliation of Riemann surfaces in \mathbb{C}^2 further denoted by $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$. For example, consider the Riemann surface

$$S_1 = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}.$$

As we are going to see in the next section 2, the surface S_1 is tangent to the complex line field $F_{a,\varepsilon}$ for any value of the parameters a and ε so it is a leaf of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$. Fix the unit circle $\delta_0 = S_1 \cap \mathbb{R}^2$. Notice, that in the case of real a and ε the phase curves of (2) restricted to \mathbb{R}^2 are topologically either lines or circles, i.e. curves with either a trivial or a non-trivial (isomorphic to \mathbb{Z}) fundamental group. For example, δ_0 is such a circular phase curve. This simple observation leads us to the definition of a marked complex cycle.

Definition 1. *A marked complex cycle of a complex foliation is a nontrivial element of the fundamental group of a leaf from the foliation with a marked base point.*

We denote a marked complex cycle by (Δ, q) where Δ is the homotopy class of loops on the leaf, all passing through the same base point q . Each loop from Δ will be called a *representative* of the cycle. In general, a real phase curve of a polynomial vector field in \mathbb{R}^2 extends to a Riemann surface tangent to the vector field's complexification in \mathbb{C}^2 . Thus, a closed phase curve in \mathbb{R}^2 defines a loop on the corresponding complex leaf, giving rise to a nontrivial element from the fundamental group of that leaf [12]. In other words, a real closed phase curve is a marked complex cycle on its complexification. As an illustration, the leaf S_1 is the complexification of the real trajectory δ_0 . The surface S_1 is topologically a cylinder and δ_0 is a nontrivial loop on it. Denoting by q_0 the point $(1, 0) \in S_1$ and by Δ_0 the homotopy class of δ_0 relative to q_0 we obtain a marked complex cycle (Δ_0, q_0) of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$.

When $\varepsilon = 0$ the line field (2) will be denoted by F_0 and its corresponding foliation by $\mathcal{F}_0(\mathbb{C}^2)$. From now on, we are going to refer to $\mathcal{F}_0(\mathbb{C}^2)$ as the *integrable foliation* and to $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ as the *perturbed foliation*. Notice that $\mathcal{F}_0(\mathbb{C}^2)$ consists of algebraic leaves of the form $S_c = \{(x, y) \in \mathbb{C}^2 \mid H(x, y) = c\}$ embedded in \mathbb{C}^2 , where $c \in \mathbb{C}$. All leaves with $c \neq 0$ are topological cylinders. Our basic approach will be to study the complex cycles of the more complicated $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ by taking advantage of the simplicity of $\mathcal{F}_0(\mathbb{C}^2)$.

One of the very useful tools for converting some of the topological properties of the foliation into dynamical properties of a holomorphic map of complex dimension one is the so called *Poincaré displacement map* [12], [13]. Next, we present a

construction of it in the case of example (2). Let T' be a complex segment (a small disc on a complex line in \mathbb{C}^2) passing through q_0 and transverse to the surface S_1 . Consider an annular neighborhood $A(\delta_0)$ of δ_0 on the surface S_1 . Next, take a tubular neighborhood $N(\delta_0)$ of $A(\delta_0)$ in \mathbb{C}^2 . It is diffeomorphic to a direct product $A(\delta_0) \times \mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc. Let ϱ be the projection of $N(\delta_0)$ onto $A(\delta_0)$ with respect to that direct product structure. Without loss of generality, we can think that $T' = \varrho^{-1}(q_0)$. Let $T \subset T'$ be a small enough open neighborhood of q_0 in T' . Take any point $q \in T$ and consider the leaf $L_{a,\varepsilon}(q)$ from the foliation $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ that passes through q . Starting from $q \in L_{a,\varepsilon}(q)$, lift the loop δ_0 to the unique path on $L_{a,\varepsilon}(q)$ that covers δ_0 under the projection ϱ . The second end-point of this lift is again on T' and we denote it by $P_{a,\varepsilon}(q)$. As a result, we obtain a one-to-one correspondence $P_{a,\varepsilon} : T \rightarrow T'$ which, by the analytic dependence of the leaves of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ on initial conditions [13], is a holomorphic map. In addition, notice that $P_{a,\varepsilon}(q_0) = q_0$ for all a and ε .

Observe that by construction, if we consider another loop $\delta'_0 \subset S_1$ passing thorough q_0 and homotopic to δ_0 on S_1 then the Poincaré map with respect to δ'_0 will be identical to $P_{a,\varepsilon}$, possibly on a smaller cross section T . This is because the homotopy between δ_0 and δ'_0 can be lifted to a homotopy on any leaf $L_{a,\varepsilon}(q)$ passing close enough to S_1 . Therefore, the endpoints on T' of the lifts of δ_0 and δ'_0 on $L_{a,\varepsilon}(q)$ will be the same. Similarly, $P_{a,\varepsilon}$ does not depend on the choice of a product structure on $N(\delta_0)$. In fact, by the tubular neighborhood theorem [10] any two product structures on $N(\delta_0)$ are isotopic via an isotopy of $N(\delta_0)$ that fixes $A(\delta_0)$ point-wise. Therefore, as a point-set, the lift of δ_0 on any near-by leaf with respect to a projection from another product structure will be the same as the lift obtained via ϱ . The difference will be only in the parametrization of the lift.

The Poincaré map $P_{a,\varepsilon}$ has the property that if two points from the cross-section T are in the same orbit of the map then they belong to the same leaf of the foliation. Moreover, a marked complex cycle of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$, with a base point on T and a representative in $N(\delta_0)$ that covers m times the loop δ_0 gives rise to an m -periodic orbit of $P_{a,\varepsilon}$. The converse is also true [15], [16]. An m -periodic orbit corresponds to a marked complex cycles of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ with a base point on T and a representative in $N(\delta_0)$ that covers δ_0 a number of m times.

Definition 2. *A marked cycle of (2) that corresponds to an m -periodic orbit of $P_{a,\varepsilon}$ is called an m -fold cycle. Whenever $m > 1$ and we do not want to specify the number m , we call the m -fold cycle a multi-fold cycle.*

Notice that whenever an m -fold cycle of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ corresponds to an m -periodic orbit of $P_{a,\varepsilon}$, the cycle also gives rise to m fixed points of the iterated map $P_{a,\varepsilon}^m$.

Definition 3. *An m -fold limit cycle of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ is an m -fold cycle that corresponds to an isolated fixed point of $P_{a,\varepsilon}^m$.*

The case $m = 1$ has been extensively studied. In fact, the real cycles of a planar polynomial line field of the form (1) extend to 1-fold cycles of its complexification. The aforementioned infinitesimal Hilbert's 16th problem [2], [12] treats exactly the special case $m = 1$. The following classical result, known as Pontryagin's criterium [17] can be stated in the following form.

Pontryagin's Theorem. *Let δ_c be an analytic family of simple closed curves on the corresponding leaves S_c of foliation (1) when $\varepsilon = 0$. Consider the analytic*

function $I(c) = \int_{\delta_c} \omega$. If there exists a value c_0 such that $I(c_0) = 0$ and $I'(c_0) \neq 0$ then there exists a continuous family δ_ε of loops, each representing a 1-fold complex limit cycle of (1). Moreover, for ε close to 0, the loops δ_ε always stay close to δ_{c_0} and $\delta_\varepsilon \rightarrow \delta_{c_0}$ as $\varepsilon \rightarrow 0$.

In contrast to 1-fold cycles, little is known about multi-fold ones. That is why, the goal of this article is to shed some light on the case $m > 1$. During a series of informal discussions, Y. Ilyashenko proposed the following questions in the spirit of Petrovskii and Landis' works [15] and [16]:

Q1. *Are there polynomial families of form (1) with Poincaré maps that have isolated periodic orbits of arbitrary period $m > 1$?*

Q2. *If $m > 1$, what may happen to an m -fold limit cycle when ε approaches 0?*

Q3. *Does a multi-fold limit cycle settle on a leaf of $\mathcal{F}_0(\mathbb{C}^2)$ as $\varepsilon \rightarrow 0$?*

For the rest of this article we try to give some answers to Ilyashenko's questions posed above for the particular family (2). Loosely stated, the main statement of the current paper is the following:

Main Result. *Multi-fold limit cycles of all possible periods appear in the family (2) when the complex parameters a and ε are chosen appropriately. Moreover, each of these cycles extends to a continuous family with respect to ε . Finally, when ε tends to zero, the multi-fold limit cycles from the family escape from a very large open subdomain of \mathbb{C}^2 that contains the surface S_1 .*

The precise formulation of the claim above will be stated in the next section as theorem 1.

The proof of the main result starts with a fairly explicit construction of the Poincaré map $P_{a,\varepsilon}$. After that, it is established that periodic orbits of all periods of $P_{a,\varepsilon}$ bifurcate from the fixed point q_0 . This immediately yields that m -fold limit cycles of all possible $m \in \mathbb{N}$ bifurcate from the cycle (Δ_0, q_0) of the family (2). For the second part of the statement, we exploit more thoroughly the connection between the topological properties of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ and the dynamics of the Poincaré map $P_{a,\varepsilon}$. First, we construct a very large smooth surface transverse to the foliation $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$. Then, we extend the Poincaré map $P_{a,\varepsilon}$ on this cross-section. We call it a *non-local Poincaré map*. Each multi-fold cycle of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ generated in the first part of our main result corresponds to a periodic orbit of $P_{a,\varepsilon}$ and together with that, determines a well-defined free homotopy class of loops in an open fibred subdomain of \mathbb{C}^2 . The topology and the fiber structure of this subdomain comes from $\mathcal{F}_0(\mathbb{C}^2)$. Moreover, as the cross-section surface is transverse to $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$, we can induce a complex structure on it so that $P_{a,\varepsilon}$ is holomorphic. Finally, the construction of the non-local complex analytic Poincaré map allows us to establish that the behavior of a multi-fold limit cycle is quite different from the behavior of a 1-fold limit cycle as ε tends to zero. By Pontryagin's theorem, the latter always stays close to some cycle from $\mathcal{F}_0(\mathbb{C}^2)$ and converges to it as ε converges to zero. In contrast to the behavior of a 1-fold limit cycle, a multi-fold one tends to escape from a very large domain in \mathbb{C}^2 when ε approaches 0. We call this phenomenon a *rapid evolution of the multi-fold limit cycle*.

The occurrence of quick escape of cycles is not that surprising if one recalls the dynamics of holomorphic maps with parabolic fixed points and their perturbations [3], [14]. In our case, $P_{a,\varepsilon}$ is a two parameter perturbation of the identity map. It is

well known that as $P_{a,\varepsilon}$ approaches the identity its m -periodic orbits, for $m > 1$, leave the map's domain. Since a periodic point of $P_{a,\varepsilon}$ represents a multi-fold cycle of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$, the multi-fold cycles should escape too. The challenge in our study is to establish the existence of periodic orbits of period $m > 1$ for the Poincaré map $P_{a,\varepsilon}$ of the foliation $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ and to extend the map's domain as much as possible. For that reason we need to analyze the link between the foliation and the map as well as to explore the topology of $\mathcal{F}_0(\mathbb{C}^2)$ globally which makes the problem quite more interesting.

So far, the third question from the list above stays unanswered. The information we have on rapid evolution reveals an interesting insight. If the answer to that question is positive, then before a multi-fold limit cycle can reach an algebraic leaf as $\varepsilon \rightarrow 0$, its representatives should change their topological properties somewhere along the way. This means that there is a possibility that the cycle settles on a critical leaf of $\mathcal{F}_0(\mathbb{C}^2)$ or goes through one or several critical leaves of $\mathcal{F}_0(\mathbb{C}^2)$, settling on a regular leaf. Since (1) is polynomial, it extends to a foliation on \mathbb{CP}^2 . Thus, another possibility is an interaction with the line at infinity.

We finish this section with a discussion about another interesting and important issue, related to question 1 above. A central problem in the study of multi-fold limit cycles is their existence in families of polynomial foliations of the form (1). Ideally, one would like to establish existence of multi-fold limit cycles in general families of type (1). Heuristically, we can follow the following steps. Using Pontryagin's theorem, we could find a family of 1-fold cycles which gives a family of isolated fixed points for the corresponding Poincaré map P_ε . For infinitely many values of ε in any neighborhood of 0, the derivative of P_ε evaluated at the fixed point will be an m -th root of unity. Thus, for such ε a local continuous family of m -periodic isolated orbits could bifurcate from the fixed point. This will happen as long as some of the resonant terms of the map's normal form do not vanish, i.e. the map is not analytically equivalent to a rotation. Since having all zero resonant terms is an extremely special property for maps with root-of-unity multiplier, we can expect that the Poincaré transformations for most foliations of the form (1) will have a lot of isolated periodic orbits and thus, the foliations themselves will have many multi-fold limit cycles. The only obstacle in this strategy is the verification that some of the resonant term coefficients of the map's normal form are nonzero. This fact imposes a challenge since the connection between the polynomial foliation and its Poincaré transformation is implicit and indirect.

2. THE MAIN THEOREM

Before giving a precise statement of the main result of the paper, we are going to fix some notations and give some definitions.

Let us verify that both $S_0 = \{(x, y) \in \mathbb{C}^2 \mid H(x, y) = 0\}$ and $S_1 = \{(x, y) \in \mathbb{C}^2 \mid H(x, y) = 1\}$ are leaves of the foliation $\mathcal{F}_{a,\varepsilon}$ for any value of the parameters a

and ε . Having in mind that $dH \wedge dH = 0$, consider the wedge product

$$\begin{aligned}
& (dH + \varepsilon(H-1)(y dx - x dy) + \varepsilon a y dH) \wedge dH = \\
& = dH \wedge dH + \varepsilon(H-1)(y dx - x dy) \wedge dH + \varepsilon a y dH \wedge dH = \\
& = \varepsilon(H-1)(y dx - x dy) \wedge dH = \\
& = 2\varepsilon(H-1)(y dx - x dy) \wedge (x dx + y dy) = \\
& = 2\varepsilon(H-1)H dx \wedge dy.
\end{aligned} \tag{3}$$

Since $H = 0$ on S_0 and $H = 1$ on S_1 , the wedge product (3) becomes zero when restricted to either S_0 or S_1 , hence both of them are tangent to the complex line-field $F_{a,\varepsilon}$, which implies that both of them are leaves of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ for any a and ε .

Look at the polynomial $H = x^2 + y^2$ as a map $H : \mathbb{C}^2 \rightarrow \mathbb{C}$. Consider the punctured plane of regular values $B = \mathbb{C} - \{0\}$ and its preimage $E = H^{-1}(B)$. Clearly, E is just \mathbb{C}^2 with the critical level set S_0 of H removed. Recall $1 \in B$ and hence $S_1 \subset E$, which is a topological cylinder (or a twice punctured sphere if you prefer).

Observe that every leaf of $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ different from S_0 is entirely contained in the domain E . Denote by $\mathcal{F}_{a,\varepsilon}$ the foliation $\mathcal{F}_{a,\varepsilon}(\mathbb{C}^2)$ with the leaf S_0 removed. Then $\mathcal{F}_{a,\varepsilon}$ is a foliation without singularities in E . In particular, when $\varepsilon = 0$ the restricted foliation \mathcal{F}_0 consists of all fibers of H with the exception of the critical one $S_0 = H^{-1}(0)$.

Let $0 < \rho_0 < R_0$, thinking of ρ_0 as very small and R_0 as large. Define the annulus $A_0 = \{c \in \mathbb{C} \mid \rho_0 < |c| < R_0\}$. Consider the preimage $E_0 = H^{-1}(A_0)$. Then, the set E_0 is the large open subdomain of \mathbb{C}^2 from which the multi-fold cycles are going to escape, according to our main statement.

Next, we look at multi-fold cycles from a topological point of view rather than dynamically.

Definition 4. A loop contained in E is called m -fold vertical provided that it is free homotopic to δ_0^m inside the domain E . A marked complex cycle of $\mathcal{F}_{a,\varepsilon}$ is called m -fold vertical provided that it has an m -fold vertical representative contained in E .

As one would expect, if a marked complex cycle of $\mathcal{F}_{a,\varepsilon}$ has at least one m -fold vertical representative in E , then all of its representatives are m -fold vertical. Indeed, let δ and δ' be two loops from the same marked cycle of $\mathcal{F}_{a,\varepsilon}$ and let δ be free-homotopic in E to δ_0^m . Then both δ and δ' are homotopic on the same leaf of $\mathcal{F}_{a,\varepsilon}$ which, in its own turn, is contained entirely in E . Therefore, both representatives are homotopic to each other inside E and since δ is free-homotopic in E to δ_0^m , so is δ' .

More interesting is the question whether the number m is a topological invariant of an m -fold vertical cycle. Assume $\delta \subset E$ is a loop representing some m -fold vertical cycle of $\mathcal{F}_{a,\varepsilon}$. Also, assume that δ is free homotopic in E to another loop $\delta'_0 \subset S_1$. As both δ_0^m and δ'_0 belong to the cylinder S_1 , whose fundamental group is \mathbb{Z} , the closed curve δ'_0 should be free homotopic on S_1 to δ^k , for some $k \in \mathbb{Z}$. Therefore, the representative δ is simultaneously m -fold and k -fold vertical. Later, in proposition 1, we are going to verify that δ_0 is not null-homotopic in E and $k = m$ always.

The leaves of foliation $\mathcal{F}_{a,\varepsilon}$, given by the line field (2), depend analytically on the two parameters a and ε . In order to study the phenomenon of rapid evolution, we need to define continuous dependance of marked limit cycles on parameters.

Definition 5. A family $\{(\Delta_\varepsilon, q_\varepsilon)\}_\varepsilon$ of marked limit cycles of $\mathcal{F}_{a,\varepsilon}$ is called continuous with respect to ε provided that there exists a continuous family of loops $\{\delta_\varepsilon\}_\varepsilon$ such that:

- a) for each ε , the closed curve δ_ε belongs to the class Δ_ε ;
- b) the base point q_ε varies continuously with respect to ε .

Let $D_r(0) = \{\varepsilon \in \mathbb{C} : |\varepsilon| < r\}$ for $r > 0$. We claim that as long as $r > 0$ is chosen small enough, rapid evolution of marked complex cycles occurs in the following form:

Theorem 1. For the two-parameter family of foliations $\mathcal{F}_{a,\varepsilon}$ given by (2) the following statements hold:

1. For any $m \in \mathbb{N}$ large enough there exists a complex parameter ε_m near $\frac{1}{m}$ and a parameter a_m such that for all ε in a neighborhood of ε_m , the polynomial foliation (2) has an m -fold vertical limit cycle with a representative inside the domain $E_0 \in \mathbb{C}^2$.
2. Furthermore, there is a parameter disc $D_{r(m)}(0)$ containing ε_m such that for any simple curve $\eta \subset D_{r(m)}(0)$ connecting ε_m to 0 there exists a relatively open subset σ of η , such that the m -fold cycle from point 1 extends on σ to a continuous family $\{(\Delta_\varepsilon, q_\varepsilon)\}_{\varepsilon \in \sigma}$ of marked cycles of $\mathcal{F}_{a,\varepsilon}$.
3. Finally, as ε moves along σ towards 0, it reaches a value $\varepsilon^* \in \sigma$ such that for any $\varepsilon \in \sigma$ past ε^* no m -fold vertical representative of $(\Delta_\varepsilon, q_\varepsilon)$ will be contained in E_0 anymore.

3. THE LOCAL POINCARÉ MAP

We begin our investigations with the construction of the Poincaré transformation locally and the computation of some of its terms.

Define $A(\delta_0)$ as a tubular neighborhood of δ_0 on the surface S_1 and $N(\delta_0)$ as a tubular neighborhood of $A(\delta_0)$ in \mathbb{C}^2 . Let

$$\mathbb{B}_{r_0} = \{\zeta \in \mathbb{C} : |\operatorname{Im}(\zeta)| < r_0\}$$

be a an infinite horizontal band in \mathbb{C} of width r_0 and let

$$D_{r_0}(1) = \{\xi \in \mathbb{C} : |\xi - 1| \leq r_0\}$$

be the disc of radius r_0 centered at 1. Consider the map

$$f_1 : \mathbb{B}_{r_0} \times D_{r_0}(1) \rightarrow N(\delta_0) \text{ defined by } f_1 : (\zeta, \xi) \mapsto (\xi \cos \zeta, \xi \sin \zeta).$$

Without loss of generality, we can think that $f_1(\mathbb{B}_{r_0} \times D_{r_0}(1)) = N(\delta_0)$. In other words, f_1 can be thought of as the universal covering map of $N(\delta_0)$. Notice, that we also have $f_1(\mathbb{B}_{r_0} \times \{1\}) = A(\delta_0) \subset S_1$.

The pull-back $f_1^* F_{a,\varepsilon}$ on $\mathbb{B}_{r_0} \times D_{r_0}(1)$ of the line field $F_{a,\varepsilon}$ is

$$f_1^* F_{a,\varepsilon} = \ker (d(\xi^2) - \varepsilon(\xi^2 - 1)\xi^2 d\zeta + a\varepsilon \xi \sin \zeta d(\xi^2)).$$

For $0 < r_1 < 1$, define the map

$$f_2 : \mathbb{B}_{r_0} \times D_{r_1}(0) \rightarrow \mathbb{B}_{r_0} \times D_{r_0}(1) \text{ where } f_2 : (z, w) \mapsto \left(z, \frac{1}{\sqrt{1-w}}\right).$$

Composing the maps f_1 and f_2 we obtain

$$f = f_1 \circ f_2 : \mathbb{B}_{r_0} \times D_{r_1}(0) \longrightarrow N(\delta_0).$$

Then the pull-back $f^*F_{a,\varepsilon}$ is

$$f^*F_{a,\varepsilon} = \ker \left(\frac{1}{(1-w)^2} \left(dw - \varepsilon w dz + \varepsilon a \frac{\sin z}{\sqrt{1-w}} dw \right) \right)$$

and since $\frac{1}{(1-w)^2}$ is well defined and nonzero for $w \in D_{r_1}(0)$, we can cancel it out and the line field becomes

$$f^*F_{a,\varepsilon} = \ker \left(dw - \varepsilon w dz + \varepsilon a \frac{\sin z}{\sqrt{1-w}} dw \right).$$

The holomorphic function $\mu_\varepsilon(z) = e^{-\varepsilon z}$ is nonzero everywhere, so

$$\begin{aligned} f^*F_{a,\varepsilon} &= \ker \left(e^{-\varepsilon z} dw - \varepsilon w e^{-\varepsilon z} dz + \varepsilon a \frac{e^{-\varepsilon z} \sin z}{\sqrt{1-w}} dw \right) \\ &= \ker \left(d(we^{-\varepsilon z}) + \varepsilon a \frac{e^{-\varepsilon z} \sin z}{\sqrt{1-w}} dw \right) \\ &= \ker (dJ^{(\varepsilon)} + a\omega^{(\varepsilon)}) \end{aligned}$$

$$\text{where } J^{(\varepsilon)} = we^{-\varepsilon z} \text{ and } \omega^{(\varepsilon)} = \frac{e^{-\varepsilon z} \sin z}{\sqrt{1-w}} dw.$$

Our next step is to define the Poincaré transformation for the foliation $\mathcal{F}_{a,\varepsilon}$, using the local chart f on the tubular neighborhood $N(\delta_0)$ of the loop δ_0 . Denote the desired map by

$$P_{a,\varepsilon} : D_{r_1}(0) \longrightarrow \mathbb{C}.$$

We are going to explain how it is constructed.

Define the path $\hat{\delta}_0^{(m)} = \{(t, 0) \in \mathbb{B}_{r_0} \times \{0\} \mid t \in [0, 2\pi m]\}$ and whenever $m = 1$ use the notation $\hat{\delta}_0 = \hat{\delta}_0^{(1)}$. Then $f(\hat{\delta}_0^{(m)}) = \delta_0^m$. For a in a neighborhood of 0 and for an appropriate choice of the radius r_1 , the segment $\hat{\delta}_0^{(m)}$ can be lifted to a path $\delta_{a,\varepsilon}^{(m)}(u)$ on the leaf of $\mathcal{F}_{a,\varepsilon}$ passing through the point $(0, u) \in \{0\} \times D_{r_1}(0)$, so that if $pr_1 : (z, w) \mapsto z$ then $pr_1(\delta_{a,\varepsilon}^{(m)}(u)) = \hat{\delta}_0^{(m)}$. Again, as before, whenever $m = 1$ we omit the superscript (m) and we write $\delta_{a,\varepsilon}(u) = \delta_{a,\varepsilon}^{(1)}(u)$. The lift $\delta_{a,\varepsilon}(u)$ has two endpoints. The first one is $(0, u)$ and the second one we denote by $(2\pi, P_{a,\varepsilon}(u))$. When $a=0$, the map $P_{0,\varepsilon}(u)$ comes from the foliation $\mathcal{F}_{0,\varepsilon}$ which in our tubular neighborhood is given by $\ker(d(we^{-\varepsilon z}))$. Then, $\delta_{0,\varepsilon}(u) = \{(t, ue^{\varepsilon t}) : t \in [0, 2\pi]\}$ and so $P_{0,\varepsilon}(u) = e^{2\pi\varepsilon}u$. Since $\delta_{a,\varepsilon}(0) = \hat{\delta}_0$, the equality $P_{a,\varepsilon}(0) = 0$ holds for all (a, ε) . As a result, the Poincaré transformation can be written down as

$$P_{a,\varepsilon}(u) = e^{2\pi\varepsilon}u + aI(u, \varepsilon)u + a^2G(u, a, \varepsilon)u$$

and its k -th iteration can be expressed as

$$P_{a,\varepsilon}^k(u) = e^{2k\pi\varepsilon}u + aI_{(k)}(u, \varepsilon)u + a^2G_{(k)}(u, a, \varepsilon)u.$$

If $\varepsilon = \frac{i}{m}$ and after m iterations the map becomes

$$P_{a,\frac{i}{m}}^m(u) = u + aI_{(m)}\left(u, \frac{i}{m}\right)u + a^2G_{(m)}\left(u, a, \frac{i}{m}\right)u.$$

In this case, denote the lift of $\hat{\delta}_0^{(m)}$ by the simpler notation $\delta_a^{(m)}(u) = \delta_{a,\frac{i}{m}}^{(m)}(u)$.

In order to study the periodic orbits of $P_{a,\varepsilon}(u)$, we are going to look at the difference $P_{a,\frac{i}{m}}^m(u) - u$. Since $(dJ^{(i/m)} + \omega^{(i/m)})|_{\delta_a^{(m)}(u)} = 0$, it can be concluded that

$$\begin{aligned} \int_{\delta_a^{(m)}(u)} (dJ^{(i/m)} + a\omega^{(i/m)}) &= 0 \quad \text{and hence} \\ \int_{\delta_a^{(m)}(u)} dJ^{(i/m)} &= -a \int_{\delta_a^{(m)}(u)} \omega^{(i/m)}. \end{aligned}$$

The one-form $dJ^{(i/m)}$ is exact and yields

$$\begin{aligned} P_{a,i/m}^m(u) - u &= P_{a,i/m}^m(u)e^{-2\pi} - ue^0 \\ &= J^{(i/m)}(2\pi m, u) - J^{(i/m)}(0, u) \\ &= \int_{\delta_a^{(m)}(u)} dJ^{(i/m)} \\ &= -a \int_{\delta_a^{(m)}(u)} \omega^{(i/m)}. \end{aligned} \tag{4}$$

Notice that when $a = 0$ the paths take the special explicit form

$$\delta_0^{(m)}(u) = \delta_{0,\frac{i}{m}}^{(m)}(u) = \{(t, e^{\frac{i}{m}t}u) \mid t \in [0, 2\pi m]\} \tag{5}$$

with endpoints $(0, u)$ and $(2\pi m, u)$. Divide equation (4) by a . When $a \rightarrow 0$ the limit of the left hand side of (4) is $I_{(m)}(u, i/m)u$. Moreover, $\delta_a^{(m)}(u) \rightarrow \delta_0^{(m)}(u)$ as $a \rightarrow 0$. As a result, we can conclude that

$$I_{(m)}(u, i/m)u = - \int_{\delta_0^{(m)}(u)} \omega^{(i/m)}.$$

Now, remembering that $\delta_0^{(m)}(u)$ is of the form (5) compute

$$\begin{aligned} I_{(m)}(u, i/m)u &= - \int_{\delta_0^{(m)}(u)} \frac{e^{-\frac{i}{m}z} \sin z}{\sqrt{1-w}} dw \\ &= - \int_0^{2\pi m} \frac{e^{-\frac{i}{m}t} \sin t}{\sqrt{1-ue^{\frac{i}{m}t}}} \left(\frac{i}{m}ue^{\frac{i}{m}t}\right) dt \\ &= - \frac{i u}{m} \int_0^{2\pi m} \frac{\sin t}{\sqrt{1-ue^{\frac{i}{m}t}}} dt. \end{aligned}$$

Since both sides of the equation are divisible by u ,

$$I_{(m)}(u, i/m) = - \frac{i}{m} \int_0^{2\pi m} \frac{\sin t}{\sqrt{1-ue^{\frac{i}{m}t}}} dt \tag{6}$$

To compute the integral in (6), notice that $1/\sqrt{1-w}$ is well defined and holomorphic in the disc $D_{r_1}(0) \not\ni 1$ so it expands as uniformly convergent series

$$(1-w)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} b_k w^k,$$

where $b_k = (-1)^k \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\dots(-\frac{1}{2}-(k-1))}{k!} \neq 0$. Thus,

$$\begin{aligned} \int_0^{2\pi m} \frac{\sin t}{\sqrt{1 - ue^{\frac{1}{m}t}}} dt &= \int_0^{2\pi m} \left(\sum_{k=0}^{\infty} b_k e^{i\frac{k}{m}t} u^k \right) \sin t dt \\ &= \sum_{k=0}^{\infty} b_k \left(\int_0^{2\pi m} e^{i\frac{k}{m}t} \sin t dt \right) u^k. \end{aligned} \quad (7)$$

The value of the integral depends on the coefficients of (7) which, in their own turn, depend on the integral

$$\begin{aligned} \int_0^{2\pi m} e^{i\frac{k}{m}t} \sin t dt &= \frac{1}{2i} \int_0^{2\pi m} e^{i\frac{k}{m}t} (e^{it} - e^{-it}) dt \\ &= \frac{1}{2i} \int_0^{2\pi m} (e^{i\frac{k+m}{m}t} - e^{i\frac{k-m}{m}t}) dt \end{aligned}$$

When $k \neq m$ the primitive of the function $(e^{i\frac{k+m}{m}t} - e^{i\frac{k-m}{m}t})$ under the integral is again $2\pi m$ -periodic, leading to the conclusion that the integral is zero. When $k = m$ the integral becomes

$$\begin{aligned} \int_0^{2\pi m} e^{it} \sin t dt &= \frac{1}{2i} \int_0^{2\pi m} e^{it} (e^{it} - e^{-it}) dt \\ &= \frac{1}{2i} \int_0^{2\pi m} (e^{i2t} - 1) dt \\ &= \frac{1}{(2i)^2} (e^{2it})_0^{2\pi m} - \frac{\pi m}{i} \\ &= i\pi m \end{aligned}$$

The computations above lead to

$$I_{(m)}(u, \frac{i}{m}) = -\frac{i}{m} b_m i\pi m u^m = \pi b_m u^m.$$

Finally, setting $c_m = \pi b_m \neq 0$, we can conclude that for $\varepsilon = \frac{i}{m}$, the Poincaré map takes the form

$$P_{a, \frac{i}{m}}^m(u) = u + a c_m u^{m+1} + a^2 G_{(m)}(u, a, i/m) u. \quad (8)$$

4. EXISTENCE OF MULTI-FOLD CYCLES

In this section we show how multi-fold limit cycles bifurcate from the cycle (Δ_0, q_0) , located on the leaf $S_1 = H^{-1}(1)$. Remember that $\delta_0 \subset S_1$ is the unit circle in the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$ centered at the origin. The set Δ_0 is the element of the fundamental group of S_1 determined by the loop δ_0 with a base point $q_0 = (1, 0) \in S_1$.

From the discussion in the introduction, the existence of a multi-fold limit cycle of $\mathcal{F}_{a, \varepsilon}$ follows from the existence of an isolated m -periodic orbit of the Poincaré transformation $P_{a, \varepsilon}$. We can see from the construction of the map that a representative of the cycle will be contained in the tubular neighborhood $N(\delta_0) \subset E$ and therefore free homotopic to δ_0^m in it. This fact immediately implies that the limit cycle will be m -fold vertical. Therefore, all we need to show is that $P_{a, \varepsilon}$ has an isolated m -periodic orbit.

We fix the radii $r_1 > 0, r_2 > 0$ and $\bar{r}_3 > 0$ so that for any $(a, \varepsilon) \in D_{r_2}(0) \times D_{\bar{r}_3}(0)$ the map $P_{a,\varepsilon} : D_{r_1}(0) \rightarrow \mathbb{C}$ is well defined. Let $m > 0$ be such that $i/m \in D_{\bar{r}_3}(0)$.

Lemma 4.1. *There exists ε_m near $\frac{i}{m}$ and a parameter a_m such that for all ε in a neighborhood of ε_m , the map $P_{a,\varepsilon}$ has an isolated periodic orbit of period m .*

Proof. The verification of the claim depends on four facts. Putting them together will help us determine the values of the parameters a and ε . In order to find a periodic orbit for the map $P_{a,\varepsilon}(u)$, we are going to look at the equation

$$P_{a,\varepsilon}^m(u) - u = 0. \quad (9)$$

Whenever $a \neq 0$ we can rewrite (9) in the form

$$\frac{e^{2\pi m\varepsilon} - 1}{a} u + I_{(m)}(u, \varepsilon)u + a G_{(m)}(u, a, \varepsilon)u = 0.$$

Furthermore, having in mind that $u = 0$ is always a solution of (9), we can divide by u and obtain

$$g(u, a, \varepsilon) = \frac{e^{2\pi m\varepsilon} - 1}{a} + I_{(m)}(u, \varepsilon) + a G_{(m)}(u, a, \varepsilon) = 0 \quad (10)$$

for $u \in D_{r_1}(0), a \in D_{r_2}(0) - \{0\}$ and $\varepsilon \in D_{\bar{r}_3}(0)$.

Fact 1. Let us focus on the equation

$$g\left(u, a, \frac{i}{m}\right) = I_{(m)}\left(u, \frac{i}{m}\right) + a G_{(m)}\left(u, a, \frac{i}{m}\right) = 0 \quad (11)$$

If necessary, decrease the radius $r_2 > 0$ enough so that if we set

$$\mathcal{M}(r_1, r_2) = \max \left\{ |a| \left| G_{(m)}\left(u, a, \frac{i}{m}\right) \right| : |u| = r_1 \text{ and } a \in D_{r_2}(0) \right\}$$

then $\mathcal{M}(r_1, r_2) < |c| r_1^m$. Since $I_{(m)}\left(u, \frac{i}{m}\right) = c_m u^m$, it follows that for $|u| = r_1$ and for any $a \in D_{r_2}(0)$

$$|c_m| |u|^m = |c_m| r_1^m > \mathcal{M}(r_1, r_2) \geq |a| \left| G_{(m)}\left(u, a, \frac{i}{m}\right) \right|,$$

so by Rouché's theorem [5], equation (11) has exactly k zeroes $u_1(a), u_2(a), \dots, u_m(a)$ in $D_{r_1}(0)$, counted with multiplicities.

Fact 2. Let $\mu(\varepsilon) = \min \{|e^{2\pi k\varepsilon} - 1| : 1 \leq k \leq m-1\}$. Regarded as a function, $\mu(\varepsilon)$ is continuous and $\mu(i/m) > 0$. Hence, there exists $r_3 > 0$, such that $\overline{D_{r_3}(i/m)} \subset D_{\bar{r}_3}(0)$. Moreover, there exists a constant $\mu > 0$, such that $\mu(\varepsilon) > \mu$ for any $\varepsilon \in D_{r_3}(i/m)$. If needed, decrease $r_2 > 0$ so that

$$\max \left\{ |a| \left| I_{(k)}(u, \varepsilon) + a G_{(k)}(u, a, \varepsilon) \right| : 1 \leq k \leq m-1 \right\} < \mu$$

for all $u \in D_{r_1}(0), a \in D_{r_2}(0)$ and $\varepsilon \in D_{r_3}(i/m)$.

Fact 3. Equation (10) can take the form

$$g(u, a, \varepsilon) = g\left(u, a, \frac{i}{m}\right) + \left(g(u, a, \varepsilon) - g\left(u, a, \frac{i}{m}\right)\right) = 0 \quad (12)$$

For any fixed $a \in D_{r_2}(0) - \{0\}$, fact 1 reveals that whenever $|u| = r_1$, the following inequalities hold:

$$\left|g\left(u, a, \frac{i}{m}\right)\right| \geq \left|I_{(m)}\left(u, \frac{i}{m}\right)\right| - |a| \left|G_{(m)}\left(u, a, \frac{i}{m}\right)\right| > 0.$$

Hence, $\mu_1(a) = \min \left\{ \left|g\left(u, a, \frac{i}{m}\right)\right| : |u| = r_1 \right\} > 0$. Notice, that for any nonzero $a \in D_{r_2}(0)$ one can find a radius $r_3(a) > 0$, continuously depending on a , such that

$$\max \left\{ \left|g(u, a, \varepsilon) - g\left(u, a, \frac{i}{m}\right)\right| : |u| = r_1, \varepsilon \in D_{r_3(a)}(i/m) \right\} < \mu_1(a),$$

Because of the last inequality, it follows by Rouché's theorem that equation (10) has as many solutions as equation (11). Thus, due to fact 1, (10) has exactly m solutions $u_1(a, \varepsilon), \dots, u_m(a, \varepsilon)$, counted with multiplicities. If we set

$$W = \bigsqcup_{0 \neq a \in D_{r_2}(0)} \left(\{a\} \times D_{r_3(a)}(i/m) \right),$$

then W is open and $\overline{W} \ni (0, \frac{i}{m})$.

Fact 4. Let $g_0(a, \varepsilon) = (e^{2\pi m \varepsilon} - 1) + a I_{(m)}(0, \varepsilon) + a^2 G_{(m)}(0, a, \varepsilon)$. Notice, that $g_0(0, \frac{i}{m}) = 0$ and $\frac{\partial g_0}{\partial \varepsilon}(0, \frac{i}{m}) = 2\pi m \neq 0$. Hence, by the implicit function theorem, it follows that for a possibly decreased $r_2 > 0$ there exists a holomorphic function $\chi : D_{r_2}(0) \rightarrow D_{r_3}(\frac{i}{m})$ such that $\chi(0) = \frac{i}{m}$ and $g_0(a, \chi(a)) = 0$ for all $a \in D_{r_2}(0)$. From here, we can see that the zero locus of g_0 inside the product domain $D_{r_2}(0) \times D_{r_3}(\frac{i}{m})$ is

$$Z = \{(a, \varepsilon) : g_0(a, \varepsilon) = 0\} = \{(a, \chi(a)) : a \in D_{r_2}(0)\}.$$

The set Z is relatively closed in $D_{r_2}(0) \times D_{r_3}(\frac{i}{m})$ so its complement $(D_{r_2}(0) \times D_{r_3}(\frac{i}{m})) - Z$ is open and nonempty. Therefore, $W \cap \left[(D_{r_2}(0) \times D_{r_3}(\frac{i}{m})) - Z \right] \neq \emptyset$ is open as well.

Now we are ready to complete the proof of the lemma. Let $(a_m, \varepsilon_m) \in W \cap \left[(D_{r_2}(0) \times D_{r_3}(\frac{i}{m})) - Z \right]$. Apply the results from fact 4 to obtain

$$\begin{aligned} g_0(a_m, \varepsilon_m) &= (e^{2\pi m \varepsilon_m} - 1) + a_m I_{(m)}(0, \varepsilon_m) + \\ &\quad + a_m^2 G_{(m)}(0, a_m, \varepsilon_m) \neq 0. \end{aligned}$$

Hence, the equation

$$\begin{aligned} P_{\varepsilon_m, a_m}^m(u) - u &= (e^{2\pi m \varepsilon_m} - 1)u + a_m I_{(m)}(u, \varepsilon_m)u + \\ &\quad + a_m^2 G_{(m)}(u, a_m, \varepsilon_m)u = 0 \end{aligned}$$

has $u_0 = 0$ as a simple root.

Since $(a_m, \varepsilon_m) \in W$, it follows from fact 3 that whenever $|u| = r_1$ the following inequality holds

$$\left|g\left(u, a_m, \frac{i}{m}\right)\right| \geq \mu_1(a_m) > \left|g(u, a_m, \varepsilon_m) - g\left(u, a_m, \frac{i}{m}\right)\right|$$

Therefore, by Rouché's theorem, the equation

$$a_m g(u, a_m, \varepsilon_m) = (e^{2\pi m \varepsilon_m} - 1) + a_m I_{(m)}(u, \varepsilon_m) + a_m^2 G_{(m)}(u, a_m, \varepsilon_m) = 0 \quad (13)$$

has as many solutions as

$$a_m g\left(u, a_m, \frac{i}{m}\right) = a_m I_{(m)}\left(u, \frac{i}{m}\right) + a_m^2 G_{(m)}\left(u, a_m, \frac{i}{m}\right) = 0. \quad (14)$$

By fact 1, equation (14) has m roots $u_1(a_m), \dots, u_m(a_m)$ contained in $D_{r_1}(0)$. For that reason, equation (13) has m solutions contained in $D_{r_1}(0)$. Let us denote them by $u_1(a_m, \varepsilon_m), \dots, u_m(a_m, \varepsilon_m)$. As it was established earlier, none of them is zero. For simplicity, let $u_j = u_j(a_m, \varepsilon_m)$, where $j = 1, \dots, m$.

By fact 2, for $1 \leq k \leq m-1$ and for $u \in D_{r_1}(0)$,

$$|e^{2\pi k \varepsilon_m} - 1| \geq \mu(a_m) > \mu > |a_m| |I_{(k)}(u, \varepsilon_m) + a_m G_{(k)}(u, a_m, \varepsilon_m)|.$$

Having in mind that $u_j \in D_{r_0}(0)$ and each of them is nonzero for $j = 1, \dots, m$, we estimate

$$\begin{aligned} |P_{a_m, \varepsilon_m}^k(u_j) - u_j| &= |u_j| |(e^{2\pi k \varepsilon_m} - 1) + a_m I_{(k)}(u_j, \varepsilon_m) \\ &\quad + a_m^2 G_{(k)}(u_j, a_m, \varepsilon_m)| \geq |u_j| (|e^{2\pi k \varepsilon_m} - 1| \\ &\quad - |a_m| |I_{(k)}(u_j, \varepsilon_m) + a_m^2 G_{(k)}(u_j, a_m, \varepsilon_m)|) > 0. \end{aligned}$$

For that reason, $P_{a_m, \varepsilon_m}^k(u_j) \neq u_j$ for $1 \leq k \leq m-1$. Hence, the orbit u_1, \dots, u_m consists of different points and therefore is periodic of period m in $D_{r_1}(0)$. \square

5. TOPOLOGY OF THE FIBER BUNDLE

Some of the constructions we would need in order to complete the proof of theorem 1 depend on the topology of the domain $E \subset \mathbb{C}^2$. That is why our goal is to understand it well.

We begin with the introduction of some useful notations. By $[\delta]_M$ we denote the set of all loops homotopic to a loop δ on a manifold M , all passing through a base point $x_0 \in M$. As usual, the homotopy class $[\delta]_M$ is an element of the fundamental group $\pi_1(M, x_0)$ of M . Our first step is to compute the fundamental group of the domain E .

To make our arguments more standard, we introduce new coordinates in \mathbb{C}^2 . Let $z = \frac{1}{\sqrt{2}}(x + iy)$ and $w = \frac{1}{\sqrt{2}}(x - iy)$. This is a unitary linear transformation of \mathbb{C}^2 and it preserves the standard Hermitian dot product $z_1 \bar{z}_2 + w_1 \bar{w}_2$. Therefore, the change of variables is an isometry and preserves all the metric properties of \mathbb{C}^2 . With respect to these coordinates $H = x^2 + y^2 = zw$ and $E = \{(z, w) \in \mathbb{C}^2 \mid zw \neq 0\}$. Also, remember the unit circle $\delta_0 = \{(\cos(2\pi t), \sin(2\pi t)) \mid t \in [0, 1]\}$ in x, y -coordinates. In z, w -coordinates, it takes the form $\delta_0 = \{(\frac{1}{\sqrt{2}}e^{2\pi t}, \frac{1}{\sqrt{2}}e^{-2\pi t}) \mid t \in [0, 1]\}$.

Lemma 5.1. *The open domain $E = \{(z, w) \in \mathbb{C}^2 \mid z \neq 0 \text{ and } w \neq 0\}$ deformation retracts onto the embedded torus $\mathbb{T} = \{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = \frac{1}{\sqrt{2}}\}$. Moreover, the circle δ_0 lies on \mathbb{T} and is not null-homotopic on it.*

Proof. Let $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ be the unit three-sphere in \mathbb{C}^2 and $K = \mathbb{S}^3 \cap \{(z, w) \in \mathbb{C}^2 \mid z = w = 0\}$. Then \mathbb{T} is embedded in \mathbb{S}^3 . The intersection of a complex line with \mathbb{S}^3 is always a great circle, and more precisely, a fiber of

the Hopf bundle [18]. Therefore, K is the classical Hopf link in the three-sphere, consisting of two great circles linked once. Let $M_K = \mathbb{S}^3 \setminus K$ be its complement. As $\mathbb{C}^2 \setminus \{(0, 0)\}$ deformation retracts onto \mathbb{S}^3 , we have that E deformation retracts onto M_K . In its own turn, M_K deformation retracts onto the torus \mathbb{T} , so we conclude that E deformation retracts onto \mathbb{T} .

For the second part of the lemma, notice that $\mathbb{T} = \{(\frac{1}{\sqrt{2}}e^{2\pi s_1}, \frac{1}{\sqrt{2}}e^{2\pi s_2}) \mid (s_1, s_2) \in [0, 1]^2\}$. From here, immediately follows that δ_0 lies on the surface of the torus. If we take an s_1 -circle and an s_2 -circle on \mathbb{T} as a homology basis for $H_1(\mathbb{T}, \mathbb{Z})$ then δ_0 has homology coordinates $(1, -1)$ and therefore is not null-homologous. Hence, it is not null-homotopic either. \square

Corollary 5.1. *The fundamental group of E is $\pi_1(E, q_0) = \mathbb{Z} \oplus \mathbb{Z}$ and $[\delta_0]_E \neq 1$.*

Proof. By lemma 5.1, the domain E deformation retracts onto the embedded torus \mathbb{T} which induces an isomorphism between the fundamental groups $\pi_1(E, q_0)$ and $\pi_1(\mathbb{T}, q_0) = \mathbb{Z} \oplus \mathbb{Z}$ [9]. The circle δ_0 is kept point-wise fixed by the deformation retraction so $[\delta_0]_E$ gets mapped to $[\delta_0]_{\mathbb{T}} \neq 1$. Hence $[\delta_0]_E \neq 1$. \square

Proposition 1. *Let $\delta \subset E$ be an m -fold vertical representative of an m -fold vertical cycle of $\mathcal{F}_{a,\varepsilon}$. Assume that δ is also free homotopic in E to another loop $\delta'_0 \subset S_1$. Then δ'_0 is free homotopic on S_1 to δ_0^m .*

Proof. Both δ_0^m and δ'_0 belong to the cylinder S_1 , whose fundamental group is \mathbb{Z} . That is why, the closed curve δ'_0 should be free homotopic on S_1 to δ_0^k , for some $k \in \mathbb{Z}$. All we need to show is that $m = k$. Indeed, consider the corresponding elements $[\delta_0^m]_E = [\delta_0]_E^m$ and $[\delta_0^k]_E = [\delta_0]_E^k$ form the fundamental group $\pi_1(E, q_0)$. Both loops δ_0^k and δ_0^m are free-homotopic in E to the same loop δ . Hence, they are free-homotopic to each other in E . Therefore, the elements $[\delta_0]_E^m$ and $[\delta_0]_E^k$ are conjugate in $\pi_1(E, q_0)$. Since, according to corollary 5.1, the fundamental group $\pi_1(E, q_0) = \mathbb{Z} \oplus \mathbb{Z}$ is Abelian, $[\delta_0]_E^m = [\delta_0]_E^k$ which is equivalent to $[\delta_0]_E^{m-k} = 1$ (here we use multiplicative notation). By the same corollary, we can see that $[\delta_0]_E \neq 1$ and $\pi_1(E, q_0) = \mathbb{Z} \oplus \mathbb{Z}$ is torsion-free. Therefore $m = k$. \square

The map $H : E \rightarrow B$ defined by the polynomial $H = x^2 + y^2$ is a smooth, locally trivial fiber bundle. For any regular value $c \in B$, the fibers $S_c = \{p \in \mathbb{C}^2 \mid H(p) = c\}$ are topological cylinders, diffeomorphic to each other [1], [13]. We use S_1 as a model fiber. The map $\nu : \mathbb{C} \rightarrow S_1$ given by $\nu(\zeta) = (\cos \zeta, \sin \zeta)$, for $\zeta \in \mathbb{C}$, is the universal covering map of S_1 . Define $\hat{C}_0 = \{\zeta \in \mathbb{C} \mid \frac{1}{2} < \text{Im}(\zeta) < \frac{3}{2}\}$. Then $C_0 = \nu(\hat{C}_0) \subset S_1$ is a non-trivial cylinder on S_1 such that $\delta_0 \cap \overline{C}_0 = \emptyset$.

Here are some facts about the topology of the fiber bundle $H : E \rightarrow B$. The unit circle $\gamma_0 = \{c \in \mathbb{C} \mid |c| = 1\}$ is a simple closed loop in B starting from 1, going around 0 counterclockwise and coming back to 1. The homotopy class of γ_0 with a base point 1 is the generator of the fundamental group $\pi_1(B, 1) \cong \mathbb{Z}$. For $c \in \gamma_0$ consider the fiber S_c . Then, if the parameter c starts from 1 and moves along the loop γ_0 until it comes back to 1 then the corresponding fiber S_c will also make one turn around the critical value 0 starting and ending up at S_1 . This procedure gives rise to an isotopy class of diffeomorphisms with a representative $\hat{D}_0 : S_1 \rightarrow S_1$ which is a Dehn twist. The map \hat{D}_0 can be chosen so that it twists the cylinder C_0 and is the identity on $S_1 \setminus \overline{C}_0$ [1].

Let $\pi(z) = e^{2\pi iz}$. Then $\pi : \mathbb{C} \rightarrow B$ is the universal covering map of the punctured plane B . Denote its group of deck transformations by $\Gamma = \{\gamma_0^m \mid m \in$

\mathbb{Z} and $\gamma_0(z) = z + 1$. As usual, it is isomorphic to the fundamental group of B . The vertical strip $\mathbb{B} = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq 1\}$ is a closed fundamental domain. Also, whenever we have a cartesian product $M_1 \times M_2$ of two sets, by pr_{M_j} we are going to denote the projection $pr_{M_j} : M_1 \times M_2 \rightarrow M_j$ where $pr_{M_j}(m_1, m_2) = m_j$ for $j = 1, 2$.

The next lemma shows that we can "unfold" the bundle $H : E \rightarrow B$ into a trivial covering bundle $pr_{\mathbb{C}} : \mathbb{C} \times S_1 \rightarrow \mathbb{C}$. Moreover, we can do so by making sure the deck group $\hat{\Gamma}$ acts in a very special manner. It not only takes vertical fibers $\{z\} \times S_1$ to vertical fibers, but what really important is that it also takes horizontal fibers $\mathbb{C} \times \{p\}$ to horizontal fibers.

Lemma 5.2. *There is a smooth covering map $\Pi : \mathbb{C} \times S_1 \rightarrow E$ with the following properties:*

1. *If $pr_{\mathbb{C}} : \mathbb{C} \times S_1 \rightarrow \mathbb{C}$ is the projection $(z, p) \mapsto z$ then $H \circ \Pi = \pi \circ pr_{\mathbb{C}}$. In other words, apart from being a covering map, Π is also a bundle map.*
2. *The deck group of $\Pi : \mathbb{C} \times S_1 \rightarrow E$ is*

$$\hat{\Gamma} = \langle (z, p) \mapsto (\gamma_0(z), D_0(p)) \rangle,$$

where $\gamma_0(z) = z + 1$ is the earlier described generator of Γ and the map $D_0 = \tilde{D}_0^{-1}$ is the Dehn twist of the cylinder $c_0 \subset S_1$ and identity everywhere else on the surface S_1 . Thus, the factor bundle $(\mathbb{C} \times S_1)/\hat{\Gamma}$ is diffeomorphically isomorphic to the bundle E .

Proof. Consider the pullback of the bundle $H : E \rightarrow B$ over the plane \mathbb{C} under the covering map π . To carry out this construction, first define the total space $\pi^*E = \{(z, q) \in \mathbb{C} \times E \mid \pi(z) = H(q)\}$. Then, the restricted projection $\kappa = (pr_{\mathbb{C}})|_{\pi^*E} : \pi^*E \rightarrow \mathbb{C}$ gives us the desired pullback bundle. Also, there is a map $\tilde{\Pi}' = (pr_E)|_{\pi^*E} : \pi^*E \rightarrow E$ that satisfies the condition $H \circ \tilde{\Pi}' = \kappa \circ \pi$ and so it is a bundle map over the map π . Together with that, $\tilde{\Pi}' : \pi^*E \rightarrow E$ is a covering map.

Because \mathbb{C} is contractible, the pullback bundle $\kappa : \pi^*E \rightarrow \mathbb{C}$ is trivializable, i.e. there is a smooth bundle isomorphism $\varsigma : \mathbb{C} \times S_1 \rightarrow \pi^*E$ so that $\kappa \circ \varsigma = pr_{\mathbb{C}} \circ id_{\mathbb{C}}$ where $id_{\mathbb{C}}$ is the identity map on \mathbb{C} . Then, the composition $\tilde{\Pi} = \tilde{\Pi}' \circ \varsigma : \mathbb{C} \times S_1 \rightarrow E$ satisfies the condition $H \circ \tilde{\Pi} = \pi \circ pr_{\mathbb{C}}$ and therefore is a bundle map and a covering map at the same time. Without loss of generality we can think that $\tilde{\Pi}(0, p) = p$, that is we identify the fiber $\{0\} \times S_1$ with the surface S_1 , where $\pi(0) = 1$.

We are going to look at the deck group $\tilde{\Gamma}$ of the covering map $\tilde{\Pi}$. For any $\tilde{\gamma} \in \tilde{\Gamma}$ we have the relation $pr_{\mathbb{C}} \circ \tilde{\gamma} = \gamma_0^m \circ pr_{\mathbb{C}}$ for some $m \in \mathbb{Z}$. That is why, just like Γ , the group $\tilde{\Gamma}$ is free Abelian with one generator $\tilde{\gamma}_0(z, p) = (\gamma_0(z), \psi(z, p))$, where $(z, p) \in \mathbb{C} \times S_1$. The map $\psi : \mathbb{C} \times S_1 \rightarrow S_1$ is smooth and if we use the notation $\psi_z(p) = \psi(z, p)$, then for any fixed $z \in \mathbb{C}$ the resulting map $\psi_z : S_1 \rightarrow S_1$ is a diffeomorphism on the standard fiber S_1 . If we factor $\mathbb{C} \times S_1$ by the action of the deck group $\tilde{\Gamma}$ we obtain the manifold $(\mathbb{C} \times S_1)/\tilde{\Gamma}$ which is isomorphic to E as a fiber bundle over B .

Consider a thin open strip $N_\epsilon = \{z \in \mathbb{C} \mid |\operatorname{Re}(z)| < \epsilon\}$ where $\epsilon < \frac{1}{3}$. Let $N'_\epsilon = \gamma_0(N_\epsilon)$ and take $\tilde{\mathbb{B}} = \mathbb{B} \cup N_\epsilon \cup N'_\epsilon$. Then we can regard the smooth map $\phi_0 : N_\epsilon \times S_1 \rightarrow N'_\epsilon \times S_1$, defined by the expression $\phi_0(z, p) = \tilde{\gamma}_0(z, p) = (\gamma_0(z), \psi(z, p))$ for any $(z, p) \in N_\epsilon \times S_1$, as a gluing map. In other words, since $\tilde{\gamma}_0$ respects the bundle structure of $\mathbb{C} \times S_1$, the quotients $(\tilde{\mathbb{B}} \times S_1)/\phi_0$ and $(\mathbb{C} \times S_1)/\tilde{\Gamma}$ are smoothly

isomorphic as fiber bundles over B (for isotopies of gluing maps, see for example [10]). Therefore, $(\tilde{\mathbb{B}} \times S_1)/\phi_0$ and E are smoothly isomorphic as bundles over B .

The strip N_ϵ deformation retracts onto the point $0 \in N_\epsilon$. Therefore, there exists a smooth deformation retraction $\vartheta : N_\epsilon \times [0, \frac{1}{2}] \rightarrow N_\epsilon$. Then, for $t = 0$ the map $\vartheta_t : N_\epsilon \rightarrow N_\epsilon$ is the identity on N_ϵ , for $t = 1$ it is the constant map $\vartheta_1 \equiv 0$ and for all $t \in [0, \frac{1}{2}]$ we have that $\vartheta_t(0) = 0$. With the help of ϑ_t , we define the isotopy

$$\begin{aligned} \phi : N_\epsilon \times S_1 \times [0, 0.5] &\rightarrow N'_\epsilon \times S \\ \phi_t(z, p) &= (\gamma_0(z), \psi(\vartheta_t(z), p)). \end{aligned}$$

As a result, when $t = 0$ we have the earlier defined map ϕ_0 . Moreover, when $t = 0.5$ we obtain the map $\phi_{0.5}(z, p) = (\gamma_0(z), \psi(0, p))$ for $(z, p) \in N_\epsilon \times S_1$. Notice that the second component of $\phi_{0.5}$ does not depend on the variable z but only on p . Whenever $z = 0$, the map $\psi_0(p) = \psi(0, p)$ is isotopic to the Dehn twist $D_0 = \tilde{D}_0^{-1}$. This follows from Picard-Lefschetz's theory as discussed previously in the current section and in [1]. Thus, we can extend smoothly the isotopy ϕ_t for $t \in [0, 1]$ so that for $t = 0$ the gluing map is ϕ_0 , for $t = 0.5$ the map becomes $\phi_{0.5}$ from above and finally when $t = 1$ we obtain $\phi_1(z, p) = (\gamma_0(z), D_0(p))$ for all $(z, p) \in N_\epsilon \times S_1$.

Notice that ϕ_t respects the vertical fibers $\{z\} \times S_1$, that is the isotopy takes place only with respect to the second coordinate, along the surface S_1 , while the first coordinate is kept the same. Therefore, $(\tilde{\mathbb{B}} \times S_1)/\phi_0$ and $(\tilde{\mathbb{B}} \times S_1)/\phi_1$ are smoothly isomorphic as fiber bundles over B . As we already saw, $(\tilde{\mathbb{B}} \times S_1)/\phi_0$ and E are isomorphic as well. Hence, $(\tilde{\mathbb{B}} \times S_1)/\phi_1$ and E are isomorphic as bundles over B . Since by construction $(\tilde{\mathbb{B}} \times S_1)/\phi_1$ and $(\mathbb{C} \times S_1)/\hat{\Gamma}$ are also isomorphic as bundles over B , we can conclude that there exists a smooth bundle isomorphism $\Phi : (\mathbb{C} \times S_1)/\hat{\Gamma} \rightarrow E$. If $v : \mathbb{C} \times S_1 \rightarrow (\mathbb{C} \times S_1)/\hat{\Gamma}$ is the quotient map, then it is a bundle map over the covering map π . When we compose it with Φ we obtain the desired bundle covering map $\Pi = \Phi \circ v : \mathbb{C} \times S_1 \rightarrow E$ satisfying the condition $H \circ \Pi = \pi \circ pr_{\mathbb{C}}$ and having $\hat{\Gamma}$ as its group of deck transformations. This completes the proof of the lemma. \square

The results from lemma 5.2 are a main tool in the proof of theorem 1. As it was mentioned already, a deck transformation $\hat{\gamma}_0^m(z, p) = (\gamma_0^m(z), D_0^m(p))$ from $\hat{\Gamma}$ maps not only vertical fibers $\{z\} \times S$ to vertical fibers $\{\gamma_0^m(z)\} \times S_1$ but also horizontal fibers $\mathbb{C} \times \{p\}$ to horizontal fibers $\mathbb{C} \times \{D_0^m(p)\}$. In particular, since D_0^m acts on $S_1 - c_0$ as the identity map, whenever $p \in S_1 \setminus C_0$, the horizontal plane $\mathbb{C} \times \{p\}$ is invariant under the action of $\hat{\Gamma}$. These facts lead us to the following conclusion.

Corollary 5.2. *For $p \in S_1 \setminus C_0$, the projection $\Pi(\mathbb{C} \times \{p\}) = B_p$ is a smoothly embedded surface in E , diffeomorphic to the punctured plane B . Moreover, B_p intersects each leaf from the integrable foliation \mathcal{F}_0 transversely at a single point.*

In particular, this corollary applies to the point $q_0 = (1, 0)$. Thus, we have obtained a global cross-section B_{q_0} .

6. THE NON-LOCAL POINCARÉ MAP

For the rest of the article, we are going to fix some $m \in \mathbb{N}$ and take the parameter $a = a_m$ as in lemma 4.1. Since $a = a_m$ is fixed, from now on we are going to drop a from all notations that contain it either as a subscript or a superscript. For example, when $a = a_m$ we will write \mathcal{F}_ε instead of $\mathcal{F}_{a, \varepsilon}$, the foliation leaves will

be denoted by L_ε instead of $L_{a,\varepsilon}$ and the notation P_ε will replace the previously accepted notation $P_{a,\varepsilon}$ for the Poincaré map.

Apart from the annulus A_0 from section 2, define two more domains in B . Remember that for a pair of numbers $0 < \rho < R$ we defined the annulus $A(\rho, R) = \{c \in \mathbb{C} \mid \rho < |c| < R\}$. Fix the small positive numbers $\rho_0 > \rho'_0 > \rho'_1 > 0$ and the large ones $0 < R_0 < R'_0 < R'_1$. Recall that $A_0 = A(\rho_0, R_0)$. Denote by A'_0 and A'_1 the annuli $A(\rho'_0, R'_0)$ and $A(\rho'_1, R'_1)$ respectively. As a result we obtain three nested open sets $A_0 \subset A'_0 \subset A'_1$.

Recall that $E_0 = H^{-1}(A_0)$ and define $E'_1 = H^{-1}(A'_1)$.

Next, we lift on \mathbb{C} all annuli from the previous paragraph to obtain the three horizontal strips $\pi^{-1}(A_0)$, $\pi^{-1}(A'_0)$ and $\pi^{-1}(A'_1)$. We fix the following cross-sections in $\mathbb{C} \times S_1$:

$$\hat{A}_0 = \pi^{-1}(A_0) \times \{q_0\}, \quad \hat{A}'_0 = \pi^{-1}(A'_0) \times \{q_0\} \quad \text{and} \quad \hat{A}'_1 = \pi^{-1}(A'_1) \times \{q_0\}.$$

All of them are subsets of $\mathbb{C} \times \{q_0\}$. Let

$$\Pi_0 = \Pi|_{\mathbb{C} \times \{q_0\}} : \mathbb{C} \times \{q_0\} \longrightarrow B_{q_0}.$$

Projecting by Π_0 , let

$$A_0(q_0) = \Pi_0(\hat{A}_0), \quad A'_0(q_0) = \Pi_0(\hat{A}'_0) \quad \text{and} \quad A'_1(q_0) = \Pi_0(\hat{A}'_1),$$

all of which are subsets of the embedded in E surface B_{q_0} .

From now on, we also use the shorter notations $\hat{z} = (z, q_0) \in \mathbb{C} \times \{q_0\}$.

Consider the pulled-back foliation $\hat{\mathcal{F}}_\varepsilon = \Pi^* \mathcal{F}_\varepsilon$ on the covering space $\mathbb{C} \times S_1$. It is invariant with respect to the action of $\hat{\Gamma}$. In other words, if $\hat{\gamma} \in \hat{\Gamma}$ and $\hat{L}_\varepsilon(z, p)$ is a leaf of $\hat{\mathcal{F}}_\varepsilon$ passing through the point $(z, p) \in \mathbb{C} \times S_1$, then $\hat{\gamma}(\hat{L}_\varepsilon(z, p)) = \hat{L}_\varepsilon(\hat{\gamma}(z, p))$.

Notice that the closure of the projection $\Pi(\hat{A}'_1) = A'_1(q_0)$ is compact in E and the tangent field F_ε is transverse to the embedded cylinder $A'_1(q_0)$ for all $|\varepsilon| \leq r$, where $r > 0$ is chosen small enough. For the next lemma we need the strip $Q = \gamma_0^{-1}(\mathbb{B}) \cup \mathbb{B} \cup \gamma_0(\mathbb{B})$ consisting of three adjacent copies of the closed fundamental domain of Γ . Take $Q'_0 = \overline{Q \cap \pi^{-1}(A'_0)}$ and let $\hat{Q}'_0 = Q'_0 \times \{q_0\}$.

Lemma 6.1. *For a small enough $r > 0$ and for any $|\varepsilon| \leq r$ there exists a smooth Poincaré map $\hat{P}_\varepsilon : \hat{Q}'_0 \rightarrow \hat{A}'_1$ associated with the foliation $\hat{\mathcal{F}}_\varepsilon$ such that for any $\hat{\gamma} \in \hat{\Gamma}$ if both points \hat{z} and $\hat{\gamma}(\hat{z})$ belong to \hat{Q}'_0 then $\hat{\gamma} \circ \hat{P}_\varepsilon = \hat{P}_\varepsilon \circ \hat{\gamma}$.*

Proof. As usual, let $pr_{S_1} : \mathbb{C} \times S_1 \rightarrow S_1$ be the projection $(z, p) \mapsto p$. By continuous dependance of $\hat{\mathcal{F}}_\varepsilon$ on parameters and initial conditions, we can choose the radius r of the parameter space so that the construction that follows holds for any $|\varepsilon| \leq r$. Choose an arbitrary point $\hat{z} \in \hat{Q}'_0$. If $\hat{L}_\varepsilon(\hat{z})$ is the leaf of the perturbed foliation $\hat{\mathcal{F}}_\varepsilon$, passing through $\hat{z} = (z, q_0)$, lift the loop δ_0 to a curve $\hat{\delta}_\varepsilon(\hat{z})$ on $\hat{L}_\varepsilon(\hat{z})$ so that $\hat{\delta}_\varepsilon(\hat{z})$ covers δ_0 under the projection pr_{S_1} . Since r is chosen small enough, the lift $\hat{\delta}_\varepsilon(\hat{z})$ is contained in the domain $\pi^{-1}(A'_1) \times S_1$ and both of its endpoints are on \hat{A}'_1 . The first endpoint is $\hat{z} \in \hat{Q}'_0$ and the second we denote by $\hat{P}_\varepsilon(\hat{z}) \in \hat{A}'_1$. Thus, we obtain the correspondence $\hat{P}_\varepsilon : \hat{Q}'_0 \rightarrow \hat{A}'_1$, which is a smooth map close to identity.

By construction, the cross-section \hat{A}'_1 is $\hat{\Gamma}$ -invariant. Now, let $\hat{z} \in \hat{Q}'_0$ and assume that $\hat{\gamma}(\hat{z}) \in \hat{Q}'_0$ for some $\hat{\gamma} \in \hat{\Gamma}$. As pointed out earlier, the arc $\hat{\delta}_\varepsilon(\hat{z})$ is the lift of δ_0 on $\hat{L}_\varepsilon(\hat{z})$ under the projection pr_{S_1} . It connects the two points $\hat{z} \in \hat{Q}$ and $\hat{P}_{a,\varepsilon}(\hat{z}) \in \hat{A}'_1$. The image $\hat{\gamma}(\hat{\delta}_\varepsilon(\hat{z}))$ lies on the leaf $\hat{\gamma}(\hat{L}_\varepsilon(\hat{z})) = \hat{L}_\varepsilon(\hat{\gamma}(\hat{z}))$ and its

endpoints are $\hat{\gamma}(\hat{z}) \in \hat{Q}'_0$ and $\hat{\gamma}(\hat{P}_\varepsilon(\hat{z})) \in \hat{A}'_1$. Since $\hat{\Gamma} \cong \mathbb{Z}$, its element $\hat{\gamma} = \hat{\gamma}_0^k$ for some $k \in \mathbb{Z}$. Therefore $pr_{S_1} \circ \hat{\gamma}(z, p) = pr_{S_1}(\gamma_0^k(z), D_0^k(p)) = D_0^k(p) = D_0^k \circ pr_{S_1}(z, p)$ for any $(z, p) \in \mathbb{C} \times S_1$. The fact that $\hat{\delta}_\varepsilon(\hat{z})$ is the lift of δ_0 on the leaf $\hat{L}_\varepsilon(\hat{z})$ from $\hat{\mathcal{F}}_\varepsilon$ means that $pr_{S_1}(\hat{\delta}_\varepsilon(\hat{z})) = \delta_0$. Similarly, to find out what the arc $\hat{\gamma}(\hat{\delta}_\varepsilon(\hat{z}))$ is a lift of, we just have to project it onto S_1 . Using the property $pr_{S_1} \circ \hat{\gamma}_0^k = D_0^k \circ pr_{S_1}$ we conclude that $pr_{S_1} \circ \hat{\gamma}_0^k(\hat{\delta}_\varepsilon(\hat{z})) = D_0^k \circ pr_{S_1}(\hat{\delta}_\varepsilon(\hat{z})) = D_0^k(\delta_0)$. Since D_0 acts like the identity everywhere on S_1 except for the thin cylinder $C_0 \subset S_1$ and $\delta_0 \cap C_0 = \emptyset$, it immediately follows that $D_0^k(\delta_0) = \delta_0$. Therefore $\hat{\gamma}(\hat{\delta}_\varepsilon(\hat{z}))$ is the lift of δ_0 on the leaf $\hat{L}_\varepsilon(\hat{\gamma}(\hat{z}))$ under the projection pr_{S_1} . With this in mind, the endpoint $\hat{\gamma}(\hat{P}_\varepsilon(\hat{z}))$ can also be rewritten as $\hat{P}_\varepsilon(\hat{\gamma}(\hat{z}))$. Thus, we obtain the relation $\hat{\gamma} \circ \hat{P}_\varepsilon = \hat{P}_\varepsilon \circ \hat{\gamma}$. \square

Lemma 6.1 allows us to extend \hat{P}_ε from a map on \hat{Q}'_0 to a $\hat{\Gamma}$ -equivariant map on the whole cross-section $\hat{A}'_0 \subset \mathbb{C} \times \{q_0\}$. This fact makes it possible for the \hat{P}_ε to descend under the covering $\Pi_0 : \hat{A}'_0 \rightarrow A'_0(q_0)$ to a Poincaré map defined on $A'_0(q_0) \subset E$.

Corollary 6.1. *The transformation \hat{P}_ε constructed in lemma 6.1 extends to a map $\hat{P}_\varepsilon : \hat{A}'_0 \rightarrow \hat{A}'_1$ for the foliation $\hat{\mathcal{F}}_\varepsilon$ such that for any $\hat{\gamma} \in \hat{\Gamma}$ the equivariance relation $\hat{\gamma} \circ \hat{P}_\varepsilon = \hat{P}_\varepsilon \circ \hat{\gamma}$ holds.*

Proof. By construction, both \hat{A}'_0 and \hat{A}'_1 are $\hat{\Gamma}$ -invariant, that is $\hat{\gamma}(\hat{A}'_0) = \hat{A}'_0$ and $\hat{\gamma}(\hat{A}'_1) = \hat{A}'_1$ for any $\hat{\gamma} \in \hat{\Gamma}$. Since $\hat{A}'_0 = \cup_{k \in \mathbb{Z}} \hat{\gamma}_0^k(\hat{Q}'_0)$, we can define \hat{P}_ε on each piece $\hat{\gamma}_0^k(\hat{Q}'_0)$ as the conjugated map

$$\hat{\gamma} \circ \hat{P}_\varepsilon \circ \hat{\gamma}^{-1} : \hat{\gamma}_0^k(\hat{Q}'_0) \longrightarrow \hat{A}'_1.$$

By lemma 6.1, for two group elements $\hat{\gamma}_1$ and $\hat{\gamma}_2 \in \hat{\Gamma}$, the two maps $\hat{\gamma}_1 \circ \hat{P}_\varepsilon \circ \hat{\gamma}_1^{-1}$ and $\hat{\gamma}_2 \circ \hat{P}_\varepsilon \circ \hat{\gamma}_2^{-1}$ agree on the intersection $\hat{\gamma}_1(\hat{Q}'_0) \cap \hat{\gamma}_2(\hat{Q}'_0)$ whenever it is nonempty. \square

Corollary 6.2. *The transformation $\hat{P}_\varepsilon : \hat{A}'_0 \rightarrow \hat{A}'_1$ associated with the foliation $\hat{\mathcal{F}}_\varepsilon$ descends to a smooth Poincaré map $P_\varepsilon : A'_0(q_0) \rightarrow A'_1(q_0)$ for the foliation \mathcal{F}_ε under the covering bundle map $\Pi : \mathbb{C} \times S_1 \rightarrow E$. In other words, for any $\hat{z} = (z, q_0) \in \hat{A}'_0$ the relation $\Pi_0 \circ \hat{P}_\varepsilon(\hat{z}) = P_\varepsilon \circ \Pi_0(\hat{z})$ holds.*

Proof. The statement follows directly from corollary 6.1. \square

On a side note, but still worth mentioning is a fact that follows from the constructions in the proof of lemma 6.1. It is not difficult to see that the Poincaré map is not sensitive to (local) homotopies of the base loop δ_0 . In other words, if δ_0 is homotopic on S_1 to another loop δ'_0 passing through q_0 , then the two maps obtained by the lifting of δ_0 and δ'_0 onto the leaves of the foliation $\hat{\mathcal{F}}_\varepsilon$ under the projection pr_{S_1} will be equal, as long as δ'_0 is close enough to δ_0 on S_1 or the radius r is kept small enough. Thus, if we slightly wiggle δ_0 on S_1 but keep the base point q_0 fixed, the resulting Poincaré map will stay the same. Consequently, the same is true for P_ε .

7. COMPLEX STRUCTURE ON THE CROSS-SECTION

Apart from the smooth structure of a fiber bundle, the space E , being a subset of \mathbb{C}^2 , has a complex structure with respect to which the foliation \mathcal{F}_ε is holomorphic and depends analytically on the parameter ε . This fact provides the foliation with very specific properties. On the other hand, the Poincaré map $P_\varepsilon : A'_0(q_0) \rightarrow A'_1(q_0)$

associated with \mathcal{F}_ε captures some topological properties of the foliation. Since some of those properties are strongly related to the holomorphic nature of the foliation, we would like our Poincaré map to reflect the complex analyticity of \mathcal{F}_ε . So far P_ε is defined as a smooth map on a subdomain of the smooth surface $A'_1(q_0)$ and therefore our next step is to induce a complex structure on $A'_1(q_0)$ in which the Poincaré transformation is holomorphic.

Since the closure of $A'_1(q_0)$ is transverse to \mathcal{F}_ε , there is an open neighborhood of $A'_1(q_0)$ in B_{q_0} transverse to \mathcal{F}_ε . Fix $\varepsilon \in D_r(0)$. Take a point $q' \in A'_1(q_0)$ and a complex cross-section $T_{q'}$ through q' , transverse to \mathcal{F}_ε . More precisely, $T_{q'}$ is a complex segment, i.e. it lies on a complex line through q' and is a real two dimensional disc.

The fact that the foliation \mathcal{F}_ε is holomorphic and $A'_1(q_0)$ is smoothly embedded surface transverse to $\mathcal{F}_{a,\varepsilon}$ provides us with convenient holomorphic flow-box charts of \mathbb{C}^2 . A chart of this kind consists of an open neighborhood $FB(q') \subset E$ of q' and a biholomorphic map

$$\beta_{q',\varepsilon} : \mathbb{D} \times \mathbb{D} \longrightarrow FB(q')$$

with the following properties:

1. $\beta_{q',\varepsilon}(0,0) = q'$;
2. $\beta_{q',\varepsilon}(\mathbb{D} \times \{0\}) = T_{q'}$;
3. $\beta_{q',\varepsilon}(\{\zeta\} \times \mathbb{D})$ is a connected component of the intersection of $FB(q')$ with the leaf $L_\varepsilon(\beta_{q',\varepsilon}(\zeta, 0))$ through the point $\beta_{q',\varepsilon}(\zeta, 0)$ for any $\zeta \in \mathbb{D}$;
4. The portion of $A'_1(q_0)$ passing through $FB(q')$ looks like the graph of a smooth map $\alpha_{q',\varepsilon} : \mathbb{D} \rightarrow \mathbb{D}$ in the chart $\mathbb{D} \times \mathbb{D}$. In other words

$$\beta_{q',\varepsilon}^{-1}(FB(q') \cap A'_1(q_0)) = \{(\zeta, \alpha_{q',\varepsilon}(\zeta)) \in \mathbb{D} \times \mathbb{D} \mid \alpha_{q',\varepsilon} : \mathbb{D} \rightarrow \mathbb{D} \text{ is smooth}\}.$$

Denote by $U_{q'}$ the open subset $FB(q_0) \cap A'_1(q_0)$ of $A'_1(q_0)$. Let $pr_j : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$ be $pr_j(\zeta_1, \zeta_2) = \zeta_j$, where $j = 1, 2$. Define the diffeomorphism

$$\begin{aligned} \phi_{q',\varepsilon} : U_{q'} &\longrightarrow \mathbb{D} \text{ by} \\ \phi_{q',\varepsilon} : q &\longmapsto pr_1 \circ (\beta_{q',\varepsilon}^{-1})|_{U_{q'}}(q) \\ \phi_{q',\varepsilon}^{-1} : \zeta &\longmapsto \beta_{q',\varepsilon}(\zeta, \alpha_{q',\varepsilon}(\zeta)). \end{aligned}$$

Consider the family of pairs $\mathcal{A}_\varepsilon(A'_1(q_0)) = \{(U_{q'}, \phi_{q',\varepsilon}) \mid q' \in A'_1(q_0)\}$.

Lemma 7.1. *The collection of charts $\mathcal{A}_\varepsilon(A'_1(q_0))$ is a holomorphic atlas for the surface $A'_1(q_0)$ with charts depending complex-analytically on ε .*

Proof. The proof is a direct verification that the transition functions $\phi_{q_1,\varepsilon} \circ \phi_{q_2,\varepsilon}^{-1}$ of two intersecting charts are holomorphic in both the coordinate variable and the parameter ε . It comes from the fact that the charts are basically projections of the open patches $U_{q'} \subset A'_1(q_0)$ onto the holomorphic cross-sections $T_{q'}$ along the leaves of the foliation \mathcal{F}_ε . Therefor the transition transformations are going to be maps from one holomorphic cross-section to another following locally the leaves of \mathcal{F}_ε . As the leaves depend holomorphically on the initial condition and the parameter ε , we obtain the desired result. For more details, one can look at [6] or [7]. \square

The choice of complex structure on the surface $A'_1(q_0)$ is justified by the next lemma. As it turns out, the map P_ε is holomorphic in the complex structure $\mathcal{A}_\varepsilon(A'_1(q_0))$.

Lemma 7.2. *The Poincaré map $P_\varepsilon : A'_0(q_0) \rightarrow A'_1(q_0)$ from corollary 6.2 associated to the foliation \mathcal{F}_ε is holomorphic in the complex structure defined by the atlas $\mathcal{A}_\varepsilon(A'_1(q_0))$ and depends complex-analytically on the parameter ε .*

Proof. The idea is based on the heuristic arguments of the previous lemma 7.1. Restricted to an open patch $U_{q_1} \subset A'_0(q_0)$, the map P_ε sends U_{q_1} inside another open patch U_{q_2} . If we look at P_ε in the two corresponding coordinate charts, we obtain a map from the holomorphic cross-section T_{q_1} to the holomorphic cross-section T_{q_2} by following the "lifts" of the loop δ_0 on the leaves of the foliation \mathcal{F}_ε . The leaves depend holomorphically on the initial condition and the parameter ε , as well as both T_{q_1} and T_{q_2} are complex segments, so we conclude that the map is as desired. More details can be found in [6] or [7]. \square

Corollary 7.1. *The surface $\hat{A}'_1 \subset \mathbb{C} \times \{q_0\}$ has a complex atlas*

$$\mathcal{A}_\varepsilon(\hat{A}'_1) = \{(\hat{U}_{\hat{z}_0}, \hat{\phi}_{\hat{z}_0, \varepsilon}) : \hat{z}_0 \in \hat{A}'_1\},$$

such that the covering map $\Pi_0 : \hat{A}'_1 \rightarrow A'_1(q_0)$ is holomorphic with respect to the complex atlas $\mathcal{A}_\varepsilon(A'_1(q_0))$. The new atlas makes the lifted Poincaré map \hat{P}_ε holomorphic, depending complex-analytically on ε .

Proof. Since as a smooth covering map Π_0 is a local diffeomorphism, simply pull back the complex structure given by $\mathcal{A}_\varepsilon(A'_1(q_0))$ to the surface \hat{A}'_1 . \square

8. PERIODIC ORBITS AND COMPLEX CYCLES

We proceed with the study of the Poincaré maps P_ε and \hat{P}_ε . More precisely, we are interested in the relationship between their periodic orbits and the marked complex cycles of the perturbed foliation \mathcal{F}_ε .

Lemma 8.1. *Let $r > 0$ be the radius obtained in lemma 6.1 and let $\varepsilon \in D_r(0)$ be fixed. Then, the following statements are true:*

1. *If $\hat{P}_\varepsilon : \hat{A}'_0 \rightarrow \hat{A}'_1$ has an m -periodic orbit $\hat{z}_1, \dots, \hat{z}_m$ in \hat{A}'_0 , then $P_\varepsilon : A'_0(q_0) \rightarrow A'_1(q_0)$ has an m -periodic orbit q_1, \dots, q_m in $A'_0(q_0)$, where $\Pi_0(\hat{z}_j) = q_j$ for $j = 1, \dots, m$.*
2. *Moreover, for each $j = 1, \dots, m$ the foliation \mathcal{F}_ε has a marked complex cycle (Δ_j, q_j) with an m -fold vertical representative δ_j contained in $E'_1 = H^{-1}(A'_1)$.*
3. *Finally, for $j = 1, \dots, m$ the cycle (Δ_j, q_j) is m -fold vertical, i.e. every representative of the cycle is free homotopic in E to δ_0^m .*

Proof. We start with the proof of the first part. Recall that

$$\Pi_0 = \Pi|_{\hat{A}'_1} : \hat{A}'_1 \longrightarrow A'_1(q_0)$$

is the universal covering map from the band \hat{A}'_1 to the cylinder $A'_1(q_0)$ embedded in $E \subset \mathbb{C}^2$. Let $q_j = \Pi_0(\hat{z}_j) \in A'_1(q_0)$ for $j = 1, \dots, m$. Due to the conjugacy relation $P_\varepsilon \circ \Pi_0 = \Pi_0 \circ \hat{P}_\varepsilon$, the image $\{q_j\}_{j=1}^m$ of the m -periodic orbit $\{\hat{z}_j\}_{j=1}^m$ of the map $\hat{P}_\varepsilon : \hat{A}'_0 \rightarrow \hat{A}'_1$ is a periodic orbit of $P_\varepsilon : A'_0(q_0) \rightarrow A'_1(q_0)$ with possibly a smaller period. Clearly, $P_\varepsilon^m(q_1) = P_\varepsilon^m(\Pi_0(\hat{z}_1)) = \Pi_0 \circ \hat{P}_\varepsilon^m(\hat{z}_1) = \Pi_0(\hat{z}_1) = q_1$.

Assume there exists a smaller $k < m$ such that $q_1 = q_{k+1}$. Since $\Pi_0 : \hat{A}'_1 \rightarrow A'_1(q_0)$ is a covering map and $\hat{\Gamma}$ restricted to \hat{A}'_1 is the deck group of Π_0 , there

exists $\hat{\gamma} \in \hat{\Gamma}$ such that $\hat{z}_{k+1} = \hat{\gamma}(\hat{z}_1)$. On the other hand, $\hat{z}_{k+1} = \hat{P}_\varepsilon^k(\hat{z}_1)$. Thus, $\hat{P}_\varepsilon^k(\hat{z}_1) = \hat{\gamma}(\hat{z}_1)$. Applying \hat{P}_ε^k to the last equality we obtain

$$\begin{aligned}\hat{P}_\varepsilon^{2k}(\hat{z}_1) &= \hat{P}_\varepsilon^k \circ \hat{\gamma}(\hat{z}_1) \\ &= \hat{\gamma} \circ \hat{P}_\varepsilon^k(\hat{z}_1) \\ &= \hat{\gamma}^2(\hat{z}_1).\end{aligned}$$

In general, $\hat{P}_\varepsilon^{jk}(\hat{z}_1) = \hat{\gamma}^j(\hat{z}_1)$ for any $j \in \mathbb{N}$. In particular, when $j = m$ we have $\hat{z}_1 = \hat{P}_\varepsilon^{mk}(\hat{z}_1) = \hat{\gamma}^m(\hat{z}_1) = \hat{\gamma}^m(\hat{z}_1)$. The identity $\hat{z}_1 = \hat{\gamma}^m(\hat{z}_1)$ implies that $\hat{\gamma}^m$ has a fixed point in \hat{A}'_1 . As a deck group of the universal covering, $\hat{\Gamma} \cong \mathbb{Z}$ acts freely on \hat{A}'_1 , so $\hat{\gamma}^m = id_{\hat{A}'_1}$ and since $\hat{\Gamma}$ has no torsion, $\hat{\gamma} = id_{\hat{A}'_1}$. We reach the conclusion $\hat{z}_{k+1} = \hat{z}_1$ which is not true. This concludes the proof of part one.

Next, we show that the second part holds. For convenience, let $\hat{z}_{m+1} = \hat{z}_1$. Observe that since all the points $\hat{z}_1, \dots, \hat{z}_m$ belong to the same orbit of \hat{P}_ε , they lie on the same leaf $\hat{L}_\varepsilon(\hat{z}_1)$ from the foliation \mathcal{F}_ε . Let $\hat{\delta}_\varepsilon(\hat{z}_j, \hat{z}_{j+1})$, for $j = 1, \dots, m$, be the lift of δ_0 on the leaf $\hat{L}_\varepsilon(\hat{z}_1)$ so that the path $\hat{\delta}_\varepsilon(\hat{z}_j, \hat{z}_{j+1})$ covers δ_0 under the projection pr_{S_1} and connects the points \hat{z}_j and \hat{z}_{j+1} . By the construction of the map \hat{P}_ε in the proof of lemma 6.1, all paths $\hat{\delta}_\varepsilon(\hat{z}_j, \hat{z}_{j+1})$ are contained in $\pi^{-1}(A'_1) \times S_1$. Therefore, the path $\hat{\delta}_\varepsilon = \cup_{j=1}^m \hat{\delta}_\varepsilon(\hat{z}_j, \hat{z}_{j+1})$ is contained in $\pi^{-1}(A'_1) \times S_1$ and goes through the points $\hat{z}_1, \dots, \hat{z}_m$. Moreover, if we parametrize $\hat{\delta}_\varepsilon$ from \hat{z}_1 to \hat{z}_m in a direction induced by the orientation of δ_0 , its two endpoints are \hat{z}_1 and $\hat{z}_{m+1} = \hat{z}_1$, so in fact $\hat{\delta}_\varepsilon$ is a loop on the leaf $\hat{L}_\varepsilon(\hat{z}_1)$.

For each $j = 1, \dots, m$ denote by $\hat{\delta}_j$ the closed curve $\hat{\delta}_\varepsilon$ but assuming that it starts from the point \hat{z}_j . As point-sets all loops $\hat{\delta}_j$ are the same $\hat{\delta}_\varepsilon$. The only difference between them is that the parametrization for each $\hat{\delta}_j$ starts from a different point from the periodic orbit of \hat{P}_ε .

Fix some $j = 1, \dots, m$. When mapping $\hat{\delta}_j$ with Π back onto E we obtain a loop $\delta_j = \Pi(\hat{\delta}_j)$ lying on the leaf $L_\varepsilon(\Pi(\hat{z}_j)) = \Pi(\hat{L}_\varepsilon(\hat{z}_j))$ from the foliation $\mathcal{F}_{a,\varepsilon}$. Moreover, δ_j is contained in $E'_1 = \Pi(\pi^{-1}(A'_1) \times S_1)$. As discussed in [15] and [16], the loop δ_j is non trivial on $\hat{L}_\varepsilon(\hat{z}_j)$ and defines a marked complex cycle (Δ_j, q_j) .

Let $Pr_{S_1} : \mathbb{C} \times S_1 \rightarrow \{0\} \times S_1$ be the map $Pr_{S_1}(z, p) = (0, p)$. Having in mind that \mathbb{C} is contractible, Pr_{S_1} is a deformation retraction, so clearly each $\hat{\delta}_j$ is free homotopic in $\mathbb{C} \times S_1$ to $\{0\} \times \delta_0^m$. As $\Pi(\mathbb{C} \times S_1) = E$, the map Π sends this free homotopy to a free homotopy in E between $\delta_j = \Pi(\hat{\delta}_j)$ and $\delta_0^m = \Pi(0, \delta_0^m)$. Hence all loops δ_j are m -fold vertical.

The third point of the current theorem follows directly from definition 4 and the discussion in the paragraph right after it. \square

Lemma 8.2. *Let ε' belong to the parameter disc $D_r(0)$, where the radius $r > 0$ is chosen as in lemma 6.1.*

1. *If $\hat{P}_{\varepsilon'} : \hat{A}'_0 \rightarrow \hat{A}'_1$ has an isolated m -periodic orbit $\{\hat{z}_j\}_{j=1}^m$ contained in \hat{A}'_0 then $P_{\varepsilon'} : A'_0(q_0) \rightarrow A'_1(q_0)$ has an isolated m -periodic orbit $\{q_j\}_{j=1}^m$ in $A'_0(q_0)$, where $\Pi_0(\hat{z}_j) = q_j$ for $j = 1, \dots, m$.*

2. *Moreover, there exists a disk $D_{r'}(\varepsilon') \subset D_r(0)$ with a small radius $r' > 0$ such that for any embedded in $D_{r'}(\varepsilon')$ curve η' , passing through ε' , there exists a continuous family $(\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon))_{\varepsilon \in \eta'}$ of periodic orbits for the map \hat{P}_ε which for*

$\varepsilon = \varepsilon'$ becomes $\hat{z}_1, \dots, \hat{z}_m$. Furthermore, this continuous family is mapped by Π_0 to a continuous family of periodic orbits $(q_1(\varepsilon), \dots, q_m(\varepsilon))_{\varepsilon \in \eta'}$ for the transformation P_ε which for $\varepsilon = \varepsilon'$ becomes the orbit q_1, \dots, q_m .

3. If \hat{P}_ε has a continuous family of periodic orbits $\{\hat{z}_j(\varepsilon)\}_{j=1}^m$ on \hat{A}'_0 for ε varying on some curve $\tilde{\eta}$ embedded in $D_r(0)$, then the perturbed foliation \mathcal{F}_ε has continuous families of marked cycles $\{(\Delta_j(\varepsilon), q_j(\varepsilon))\}_{\varepsilon \in \tilde{\eta}}$ for $j = 1, \dots, m$ where $q_j(\varepsilon) = \Pi_0(\hat{z}_\varepsilon)$.

Proof. Point one follows from lemma 8.1 together with the conjugacy condition $\Pi_0 \circ \hat{P}_{\varepsilon'} = P_{\varepsilon'} \circ \Pi_0$. As Π_0 is locally a diffeomorphism, then the fact that $\{\hat{z}_j\}_{j=2}^m$ is isolated implies that $\{q_j\}_{j=2}^m$ is isolated too.

As $\hat{P}_{\varepsilon'}(\hat{z}_1) = \hat{z}_1$, we choose a chart $(\hat{U}_{\hat{z}_1}, \hat{\phi}_{\hat{z}_1, \varepsilon})$ from the atlas $\mathcal{A}_\varepsilon(\hat{A}'_1)$ around the point \hat{z}_1 and a smaller neighborhood $\hat{U}'_{\hat{z}_1}$ of the same point such that $\hat{U}'_{\hat{z}_1} \subset \hat{U}_{\hat{z}_1}$ and $\hat{P}_{\varepsilon'}(\hat{U}'_{\hat{z}_1}) \subset \hat{U}_{\hat{z}_1}$. Let $D' = \hat{\phi}_{\hat{z}_1, \varepsilon'}(\hat{U}'_{\hat{z}_1}) \subset \mathbb{D}$ where $\hat{\phi}_{\hat{z}_1, \varepsilon'}(\hat{z}_1) = 0 \in D'$. If $r' > 0$ is chosen small enough, then

$$\hat{P}_\varepsilon^{(m)} = \hat{\phi}_{\hat{z}_1, \varepsilon} \circ \hat{P}_\varepsilon^m \circ \hat{\phi}_{\hat{z}_1, \varepsilon}^{-1} : D' \longrightarrow \mathbb{D}$$

for $\varepsilon \in D_{r'}(\varepsilon') \subset D_r(0)$. Notice that $\hat{P}_{\varepsilon'}^{(m)}(0) = 0$. The complex valued function

$$\tilde{F} : D' \times D_{r'}(\varepsilon') \rightarrow \mathbb{C} \text{ defined as } \tilde{F}(\zeta, \varepsilon) = \hat{P}_\varepsilon^{(m)}(\zeta) - \zeta$$

is holomorphic with respect to $\zeta \in D'$ and with respect to $\varepsilon \in D_{r'}(\varepsilon')$. By Hartogs' theorem [8], \tilde{F} is holomorphic with respect to $(\zeta, \varepsilon) \in D' \times D_{r'}(\varepsilon')$. Since $\hat{P}_{\varepsilon'}^{(m)}(0) = 0$, the point $(0, \varepsilon')$ is a zero of \tilde{F} , that is $\tilde{F}(0, \varepsilon') = 0$.

Let us look at the zero locus of \tilde{F} in $D' \times D_{r'}(\varepsilon')$. The fact that the periodic orbit $\{\hat{z}_j\}_{j=1}^m$ is isolated means that \hat{z}_1 is an isolated fixed point for the map $\hat{P}_{\varepsilon'}^m$. Therefore 0 is an isolated fixed point for $\hat{P}_{\varepsilon'}^{(m)}$ and thus, it is an isolated zero for the holomorphic function $\tilde{F}(\zeta, \varepsilon')$ regarded as a function of ζ only. By Weierstrass' preparation theorem [8], [4], we can write

$$\tilde{F}(\zeta, \varepsilon) = \prod_{j=1}^s (\zeta - \alpha_j(\varepsilon)) \theta(\zeta, \varepsilon),$$

where $\theta(0, \varepsilon') \neq 0$ and $\{\alpha_j(\varepsilon) : j = 1, \dots, s\}$ depend analytically on $\varepsilon \in D_{r'}(\varepsilon')$, satisfying the conditions $\alpha_1(\varepsilon') = \dots = \alpha_s(\varepsilon') = 0$ and possibly branching into each other.

Now, let η' be some simple curve embedded in the disc $D_{r'}(\varepsilon')$ and passing through ε' . For ε varying on η' , we can choose a branch, denoted for simplicity by $\alpha_1(\varepsilon)$. Then the desired continuous family for \hat{P}_ε can be constructed by setting $\hat{z}_1(\varepsilon) = \hat{\phi}_{\hat{z}_1, \varepsilon}^{-1}(\alpha_1(\varepsilon))$ and $\hat{z}_{j+1}(\varepsilon) = \hat{P}_{a, \varepsilon}^j(\hat{z}_1(\varepsilon))$ for $j = 1, \dots, m-1$. Its image under the covering Π_0 will provide the continuous family of periodic orbits $\{q_j(\varepsilon)\}_{j=1}^m$ for P_ε .

The third point of the current statement follows directly from lemma 8.1 part two. For each $\varepsilon \in \eta'$ and $j = 1, \dots, m$ the constructed representative $\delta_j(\varepsilon) \subset E'_1$ depends in fact continuously on the parameter $\varepsilon \in \eta'$ because the leaves of all foliations we work with depend continuously on the initial points and the parameter ε . Thus the loops $\delta_j(\varepsilon)$ and the base points $q_j(\varepsilon)$, continuously depending on ε , give rise to continuous families $\{(\Delta_j(\varepsilon), q_j(\varepsilon))\}_{\varepsilon \in \eta'}$ \square

9. PROOF OF THEOREM 1

From now on we assume that $a = a_m$ as provided by lemma 4.1. Let η be an embedded in $D_r(0)$ curve, connecting ε_0 to 0. For convenience, define a natural linear order \preceq on it so that $0 \prec \varepsilon_0$. We begin this section with a summary of the proof.

First, with the help of lemma 4.1, we show that the Poincaré map $P_{a,\varepsilon}$ of the foliation $\mathcal{F}_{a,\varepsilon}$ has an isolated periodic orbit $\{q_j\}_{j=1}^m$ for the fixed parameter $a = a_m$ and ε_0 very close to $\varepsilon_m \in D_r(0)$. Then, $\{q_j\}_{j=1}^m$ can be extended to a continuous family of periodic orbits $\{q_j(\varepsilon)\}_{j=1}^m$ on $A'_0(q_0)$ for values of the parameter ε defined on an relatively open subset η_{\max} of the path η . We also find out that there exists $\varepsilon^* \in \eta_{\max}$ such that if $\varepsilon \in \eta_{\max}$ and $\varepsilon \prec \varepsilon^*$ then some $q_{j_0}(\varepsilon) \in A'_0(q_0) \setminus A_0(q_0)$. We denote $q_{j_0}(\varepsilon) = q(\varepsilon)$. By point 3 of lemma 8.2, there exists a continuous family of marked cycles $\{(\Delta(\varepsilon), q(\varepsilon))\}_{\varepsilon \in \eta_{\max}}$ defined on η_{\max} . For the value $\varepsilon = \varepsilon_0$ the cycle $(\Delta(\varepsilon_0), q(\varepsilon_0))$ is limit m -fold vertical and has a representative $\delta(\varepsilon_0)$ contained in the domain $E_0 = H^{-1}(A_0)$. As ε moves on η_{\max} towards 0, it passes through the point ε^* and as a result of this the point $q(\varepsilon)$ leaves $A_0(q_0)$ and therefore it leaves E_0 as well. Consequently, for any $\varepsilon \in \eta_{\max}$ with $\varepsilon \prec \varepsilon^*$ no representative of $(\Delta(\varepsilon), q(\varepsilon))$ is contained in E_0 because all of them pass through the same base point $q(\varepsilon)$ and $q(\varepsilon)$ is not in E_0 anymore. This last fact concludes the proof of theorem 1 with $\sigma = \eta_{\max}$.

Next, we continue with the detailed proof of the main result of this article. The Poincaré map constructed in section 3 is in fact the local Poincaré map from the introduction of the article, defined on the complex cross-section $T'_{q_0} = \{(x, 0) \in \mathbb{C}^2 \mid |x - 1| < r'_0\}$ where r'_0 is very small. We denote this local map by $P_\varepsilon^{loc} : T'_{q_0} \rightarrow T'_{q_0}$. Notice that P_ε^{loc} is constructed using the tubular neighborhood $N(\delta_0)$ together with a projection, we call $\varrho : N(\delta_0) \rightarrow A(\delta_0)$, coming from the direct product structure on $N(\delta_0)$ selected in section 3. Since the covering map $\Pi : \mathbb{C} \times S_1 \rightarrow E$ is a local diffeomorphism, the tubular neighborhood $N(\delta_0)$ inherits via Π the product structure of $\mathbb{C} \times S_1$ together with a projection ϱ' coming from the natural projection pr_{S_1} which maps $\mathbb{C} \times S_1$ onto S_1 . As already mentioned in the introduction, there exists an isotopy on $N(\delta_0)$ that keeps $A(\delta_0)$ fixed point-wise and sends one product structure to the other [10]. Therefore, the lifts of δ_0 via ϱ and ϱ' on any near-by leaf of \mathcal{F}_ε will coincide as point-sets. In conclusion, the map P_ε^{loc} is in fact a representation of the non-local Poincaré map P_ε in a complex chart from the atlas $\mathcal{A}_\varepsilon(A'_1(q_0))$ defined in section 7. Together with this, by corollary 7.1, the local map P_ε^{loc} on the cross-section T'_{q_0} is a representation of the lifted Poincaré transformation \hat{P}_ε in a complex chart from the atlas $\mathcal{A}_\varepsilon(\hat{A}'_1)$. Combining this fact with lemma 4.1, we conclude that for a choice of ε_0 near $\varepsilon_m \in D_r(0)$ the transformation \hat{P}_ε has an isolated m -periodic orbit $\{\hat{z}_j\}_{j=1}^m$. By lemma 8.1, for $j = 1, \dots, m$ there are m -fold vertical limit cycles (Δ_j, q_j) each having a representative δ_j contained in E'_1 . Comparing the construction of P_ε^{loc} with that of \hat{P}_ε and P_ε as well as having in mind the fact that the lifts are independent on the choice of a product structure on $N(\delta_0)$, one concludes that δ_j is contained in $N(\delta_0)$ which is a thin neighborhood of δ_0 and in its own turn is a subset of E_0 . Thus, the representative δ_j of the limit cycle (Δ_j, q_j) is an m -fold vertical loop contained in the domain E_0 for $j = 1, \dots, m$.

By lemma 8.2, there exists $D_{r_0}(\varepsilon_0) \subset D_r(0)$ for some $r_0 > 0$, such that if $\eta_0 = \eta \cap D_{r_0}(\varepsilon_0)$, then there is a continuous family of periodic orbits $(\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon))_{\varepsilon \in \eta_0}$ of the map \hat{P}_ε on the cross-section \hat{A}'_0 .

Define $\eta_{\max} \subseteq \eta$ as the maximal relatively open subset of η on which the continuous family $(\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon))_{\varepsilon \in \eta_{\max}}$ of periodic orbits for \hat{P}_ε exists on the cross-section \hat{A}'_0 . Since $\eta_0 \neq \emptyset$ is relatively open in η , the inclusion $\eta_0 \subseteq \eta_{\max}$ holds and therefore $\eta_{\max} \neq \emptyset$. By point 3 from lemma 8.2, for each $j = 1, \dots, m$ there exists a continuous family of marked complex cycles $\{(\Delta_j(\varepsilon), q_j(\varepsilon))\}_{\varepsilon \in \eta_{\max}}$ with $q_j(\varepsilon) = \Pi(\hat{z}_j(\varepsilon))$. Near $\varepsilon_0 \in \eta_{\max}$ the cycles $(\Delta_j(\varepsilon), q_j(\varepsilon))$ are limit and have m -fold vertical representatives $\delta_j(\varepsilon)$ contained in E_0 because when $\varepsilon = \varepsilon_0$ each cycle $(\Delta_j(\varepsilon_0), q_j(\varepsilon_0)) = (\Delta_j, q_j)$ is limit and has an m -fold vertical representative, namely $\delta_j = \delta_j(\varepsilon_0)$, contained inside the domain E_0 . We would like to find out what happens to the cycles as ε varies on η_{\max} .

Let η' be the set of all ε from η_{\max} for which the periodic orbits from the continuous family $(\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon))_{\varepsilon \in \eta_{\max}}$ are entirely contained in $\hat{A}_0 \subset \hat{A}'_0$. As we already saw, at ε_0 the orbit $\hat{z}_1(\varepsilon_0), \dots, \hat{z}_m(\varepsilon_0)$ is inside \hat{A}_0 and by continuity, the orbits $\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon)$ are also contained in \hat{A}_0 for ε near ε_0 . This fact shows that $\eta' \neq \emptyset$ and in fact it has a nonempty interior.

Let $\varepsilon^{**} = \inf_{\eta}(\eta_{\max})$ be the infimum of η_{\max} with respect to the linear ordering on η . Then, $D_{\frac{1}{N}}(\varepsilon^*) \cap \eta_{\max} \neq \emptyset$ for all $N \in \mathbb{N}$. Similarly, define $\varepsilon^* = \inf_{\eta}(\eta')$ as the infimum of η' . The inclusion $\eta' \subseteq \eta_{\max}$ implies that $\varepsilon^{**} \preceq \varepsilon^*$. We are going to show that $\varepsilon^{**} \neq \varepsilon^*$.

Assume $\varepsilon^{**} = \varepsilon^*$, that is for all $N \in \mathbb{N}$ there exists $\varepsilon_N \in D_{\frac{1}{N}}(\varepsilon^{**}) \cap \eta_{\max}$ such that $\hat{z}_1(\varepsilon_N), \dots, \hat{z}_m(\varepsilon_N)$ is contained in \hat{A}_0 . As explained in point 2 of lemma 8.2 the family of periodic orbits $(\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon))_{\varepsilon \in \eta_{\max}}$ is mapped by Π_0 to a periodic family $(q_1(\varepsilon), \dots, q_m(\varepsilon))_{\varepsilon \in \eta_{\max}}$ of the map P_ε on the surface $A'_0(q_0)$. Moreover, the corresponding orbits $q_1(\varepsilon_N), \dots, q_m(\varepsilon_N)$ are inside $A_0(q_0) \subset A'_0(q_0)$ for all $N \in \mathbb{N}$. In particular, the sequence $\{q_1(\varepsilon_N)\}_{N \in \mathbb{N}}$ is contained in the compact cylinder $\overline{A_0(q_0)}$. Then, there exists $q_1^* \in \overline{A_0(q_0)}$ and a subsequence $\{q_1(\varepsilon_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} q_1(\varepsilon_n) = q_1^*$ and $\lim_{n \rightarrow \infty} \varepsilon_n = \varepsilon^{**}$. By continuity, the identity $P_{\varepsilon_n}^m(q_1(\varepsilon_n)) = q_1(\varepsilon_n)$ converges to $P_{\varepsilon^{**}}^m(q_1^*) = q_1^*$ as $n \rightarrow \infty$. Generate a periodic orbit q_1^*, \dots, q_m^* by setting $q_{j+1}^* = P_{\varepsilon^{**}}^j(q_1^*)$ for $j = 1, \dots, m-1$. Since $q_{j+1}(\varepsilon_n) = P_{\varepsilon_n}^j(q_1(\varepsilon_n))$, the limit for each $q_j(\varepsilon_n) \in A_0(q_0)$ is $q_j^* \in \overline{A_0(q_0)}$ as $n \rightarrow \infty$. Thus, the periodic orbit q_1^*, \dots, q_m^* is the limit of the periodic orbits $q_1(\varepsilon_n), \dots, q_m(\varepsilon_n)$ and is contained in $\overline{A_0(q_0)}$.

We will show that under the current assumptions $\varepsilon^{**} = 0$. Assume that $\varepsilon^{**} \neq 0$. Then $\{\varepsilon \in \eta \mid \varepsilon \prec \varepsilon^{**}\} \neq \emptyset$. We proceed as in the proof of lemma 8.2. The point $q_1^* \in \overline{A_0(q_0)}$ is fixed by the map $P_{\varepsilon^{**}}^m$. Take a complex chart $(U_{q_1^*}, \phi_{q_1^*, \varepsilon^{**}})$ form the atlas $\mathcal{A}_{\varepsilon^{**}}(A'_1(q_0))$ around q_1^* and a smaller neighborhood $U'_{q_1^*} \subset U_{q_1^*}$ of the same point such that $P_{\varepsilon^{**}}^m(U'_{q_1^*}) \subset U_{q_1^*}$. Let $D' = \phi_{q_1^*, \varepsilon^{**}}(U'_{q_1^*, \varepsilon^{**}}) \subset \mathbb{D}$ where $\phi_{q_1^*, \varepsilon^{**}}(q_1^*) = 0 \in D'$. Choose $r^* > 0$ small enough such that

$$P_\varepsilon^{(m)} = \phi_{q_1^*, \varepsilon} \circ P_\varepsilon^m \circ \phi_{q_1^*, \varepsilon}^{-1} : D' \longrightarrow \mathbb{D}$$

for $\varepsilon \in D_{r^*}(\varepsilon^{**}) \subset D_r(0)$. Notice that $P_{\varepsilon^{**}}^{(m)}(0) = 0$. The complex valued function

$$\tilde{F} : D' \rightarrow \mathbb{C} \text{ defined as } \tilde{F}(\zeta, \varepsilon) = P_\varepsilon^{(m)}(\zeta) - \zeta$$

is holomorphic with respect to $(\zeta, \varepsilon) \in D' \times D_{r^*}(\varepsilon^{**})$. Since $P_{\varepsilon^{**}}^{(m)}(0) = 0$, the point $(0, \varepsilon^{**})$ is a zero of \tilde{F} , that is $\tilde{F}(0, \varepsilon^{**}) = 0$.

We are interested in the zero locus of \tilde{F} in the domain $D' \times D_{r^*}(\varepsilon^{**})$. If we assume for a moment that $\tilde{F}(\zeta, \varepsilon) \equiv 0$ on D' then we would have the identity $P_{\varepsilon}^{(m)}(\zeta) \equiv \zeta$ on D' and therefore $P_{\varepsilon}^m(q) \equiv q$ on the open subset $U'_{q_1^*} \subset A'_0(q_0)$. Because of the analyticity of $P_{\varepsilon}^m(q)$ with respect to both q and ε , the identity $P_{\varepsilon}^m(q) \equiv q$ will hold on all of $A'_0(q_0)$ and for all $\varepsilon \in D_r(0)$. In particular, it will be true for $\varepsilon = \varepsilon_0$. But for that value the map P_{ε}^m has an isolated fixed point $q_1(\varepsilon_0) \in A_0(q_0) \subset A'_0(q_0)$ which leads to a contradiction. Therefore \tilde{F} is not identically zero.

There are two cases for \tilde{F} . Either $\tilde{F}(\zeta, \varepsilon^{**}) \equiv 0$ or $\tilde{F}(\zeta, \varepsilon^{**}) \neq 0$ for $\zeta \in D'$. For both of those options \tilde{F} can be written as

$$\tilde{F}(\zeta, \varepsilon) = (\varepsilon - \varepsilon^{**})^b F(\zeta, \varepsilon)$$

where $F(\zeta, \varepsilon^{**}) \neq 0$ and $b \geq 0$. When $b > 0$ we have the first case and when $b = 0$ we have the second case.

Let us look at the zero locus of F . By Weierstrass' preparation theorem [4], [8], F can be written as

$$F(\zeta, \varepsilon) = \prod_{j=1}^s (\zeta - \alpha_j(\varepsilon)) \theta(\zeta, \varepsilon),$$

where $\theta(0, \varepsilon^{**}) \neq 0$ and $\{\alpha_j(\varepsilon) : j = 1, \dots, s\}$ depend analytically on ε , satisfying the equalities $\alpha_1(\varepsilon^{**}) = \dots = \alpha_s(\varepsilon^{**}) = 0$ and possibly branching into each other. Without loss of generality, we can think that D' is chosen small enough so that $\theta(\zeta, \varepsilon) \neq 0$ for all $(\zeta, \varepsilon) \in D' \times D_{r^*}(\varepsilon^{**})$. Let $\tilde{\alpha}_j(\varepsilon) = \phi_{q_1^*, \varepsilon}^{-1}(\alpha_j(\varepsilon))$. Since $q_1(\varepsilon_n) \rightarrow q_1^*$, there exists $N_0 \in \mathbb{N}$ such that $q_1(\varepsilon_n) \in U'_{q_1^*}$ for $n > N_0$. By the continuity of $q_1(\varepsilon)$, for each $\varepsilon \in D_{r^*}(\varepsilon^{**}) \cap \eta_{\max}$ we have that $q_1(\varepsilon) = \tilde{\alpha}_j(\varepsilon)$ for some $j = 1, \dots, s$. For simplicity of notation, assume $q_1(\varepsilon) = \tilde{\alpha}_1(\varepsilon)$. Thus, $q_1(\varepsilon)$ converges to q_1^* as $\varepsilon \rightarrow \varepsilon^{**}$ always staying on the zero locus of F . Therefore we can extend $q_1(\varepsilon)$ continuously on η past ε^{**} by setting $q_1(\varepsilon) = \tilde{\alpha}_1(\varepsilon)$ for $\varepsilon \in D_{r^*}(\varepsilon^{**}) \cap \{\varepsilon \in \eta : \varepsilon \preceq \varepsilon^{**}\}$. By construction, the identity $P_{\varepsilon}^m(\tilde{\alpha}_1(\varepsilon)) = \tilde{\alpha}_1(\varepsilon)$ holds and if we set $q_{j+1}(\varepsilon) = P_{\varepsilon}^j(\tilde{\alpha}_1(\varepsilon))$ we obtain a continuation of the family $q_1(\varepsilon), \dots, q_m(\varepsilon)$ on the relatively open arc $D_{r^*}(\varepsilon^{**}) \cap \{\varepsilon \in \eta : \varepsilon \preceq \varepsilon^{**}\}$. As a result we have a continuous family $(q_1(\varepsilon), \dots, q_m(\varepsilon))_{\varepsilon \in \tilde{\eta}}$ of periodic orbits of P_{ε} defined for $\varepsilon \in \tilde{\eta} = (D_{r^*}(\varepsilon^{**}) \cap \{\varepsilon \in \eta : \varepsilon \preceq \varepsilon^{**}\}) \cup \eta_{\max}$ which is relatively open in η .

Since the family $\hat{z}_1(\varepsilon), \dots, \hat{z}_m(\varepsilon)$ is the lift of $q_1(\varepsilon), \dots, q_m(\varepsilon)$ for $\varepsilon \in \eta_{\max}$ and the latter extends on $\tilde{\eta} \supset \eta_{\max}$, the former also extends on $\tilde{\eta}$ as a family of periodic orbits for \hat{P}_{ε} on the cross-section \hat{A}'_0 . This conclusion contradicts the maximality of η_{\max} , stemming from the assumption that $\varepsilon^{**} \neq 0$. Therefore $\varepsilon^{**} = 0$ and $q_1(0), \dots, q_m(0)$ is a periodic orbit of $P_0 = id_{A'_0(q_0)}$. For that reason, $q_1(0) = \dots = q_m(0) = q^*$ inside $A'_0(q_0)$.

Take a complex chart $(U_{q^*}, \phi_{q^*, 0})$ on $A'_1(q_0)$ around the point q^* and choose a smaller neighborhood $U'_{q^*} \subset U_{q^*}$ of q^* such that $P_{\varepsilon}^k(U'_{q^*}) \subset U_{q^*}$ for all $k = 1, \dots, m$ and $\varepsilon \in D_{\tilde{r}_0}(0)$, where $\tilde{r}_0 > 0$ is small enough. Let $D' = \phi_{q^*, 0}(U'_{q^*}) \subset \mathbb{D}$ and

$$P_{q^*, \varepsilon} = \phi_{q^*, \varepsilon} \circ P_{\varepsilon} \circ \phi_{q^*, \varepsilon}^{-1} : D' \longrightarrow \mathbb{D}.$$

Denote by $\zeta_j(\varepsilon) = \phi_{q^*, \varepsilon}(q_j(\varepsilon))$ for $\varepsilon \in D_{\tilde{r}_0}(0) \cap \eta_{\max} = \eta_0$ and $j = 1, \dots, m$. Then $\zeta_1(\varepsilon), \dots, \zeta_m(\varepsilon)$ is a periodic orbit for $P_{q^*, \varepsilon}$ in D' . Notice, that due to the holomorphic nature of the map P_{ε} , those $\varepsilon \in \eta_{\max}$ for which $q_i(\varepsilon) = q_j(\varepsilon)$, where $1 \leq i < j \leq m$,

are isolated because the family at ε_0 consists of an isolated m -periodic orbit. As before, $P_{q^*,\varepsilon}(\zeta)$ is holomorphic with respect to (ζ, ε) . Then we can write it as

$$P_{q^*,\varepsilon}(\zeta) = \zeta + \varepsilon^l I(\zeta) + \varepsilon^{l+1} R(\zeta, \varepsilon)$$

where $I(\zeta) \not\equiv 0$ and $l \geq 1$. If we iterate this map m times we obtain the representation

$$P_{q^*,\varepsilon}^m(\zeta) = \zeta + \varepsilon^l m I(\zeta) + \varepsilon^{l+1} R_{(m)}(\zeta, \varepsilon).$$

For $\varepsilon \in \eta_0 \setminus \{0\}$ the equations

$$\begin{aligned} P_{q^*,\varepsilon}(\zeta) - \zeta &= \varepsilon^l (I(\zeta) + \varepsilon R(\zeta, \varepsilon)) = 0 \quad \text{and} \\ P_{q^*,\varepsilon}^m(\zeta) - \zeta &= \varepsilon^l (m I(\zeta) + \varepsilon R_{(m)}(\zeta, \varepsilon)) = 0 \end{aligned}$$

are divisible by ε^l and thus, become

$$I(\zeta) + \varepsilon R(\zeta, \varepsilon) = 0 \quad \text{and} \quad m I(\zeta) + \varepsilon R_{(m)}(\zeta, \varepsilon) = 0 \quad (15)$$

The function $I(\zeta)$ is not identically zero, so it has isolated zeroes. Choose $D'' \subset D'$ to be a small closed disc centered at zero, so that no zeroes of $I(\zeta)$ are contained in $D'' \setminus \{0\}$. In particular, $I(\zeta) \neq 0$ for $\zeta \in \partial D''$. We can decrease the parameter radius $r_0 > 0$ enough so that by Rouché's theorem [5] the equations (15) will have the same number of zeroes, counting multiplicities, as the equation $I(\zeta) = 0$. Clearly, all zeroes of $P_{q^*,\varepsilon}(\zeta) - \zeta$ are zeroes of $P_{q^*,\varepsilon}^m(\zeta) - \zeta$ because the fixed points of $P_{q^*,\varepsilon}$ are fixed points of $P_{q^*,\varepsilon}^m$ but not the other way around. On the other hand, as already noted, for almost every $\varepsilon \in D_{r_0}(0)$ there is an m -periodic orbit $\zeta_1(\varepsilon), \dots, \zeta_m(\varepsilon)$ for the map $P_{q^*,\varepsilon}$ inside D'' . Thus, we can see that $P_{q^*,\varepsilon}^m(\zeta) - \zeta$ has at least m zeroes more than $P_{q^*,\varepsilon}(\zeta) - \zeta$, which contradicts the fact that both of these should have the same number of zeroes. The contradiction comes from the assumption that $\varepsilon^{**} = \varepsilon^*$. Therefore we conclude that $\varepsilon^{**} \neq \varepsilon^*$ and in fact $\varepsilon^{**} \prec \varepsilon^*$.

Let $\eta_1 = \{\varepsilon \in \eta_{\max} \mid \varepsilon^{**} \prec \varepsilon \prec \varepsilon^*\}$. Then for any $\varepsilon \in \eta_1$ at least one $\hat{z}_{j_0}(\varepsilon)$ is contained in \hat{A}'_0 but not in \hat{A}_0 . Then, its image $q(\varepsilon) = \Pi_0(\hat{z}_{j_0}(\varepsilon))$ varies continuously on $A'_0(q_0)$ with respect to $\varepsilon \in \eta_{\max}$. Moreover, when $\varepsilon \in \eta_1 \subset \eta_{\max}$ what happens is that $q(\varepsilon) \in A'_0(q_0) \setminus A_0(q_0)$. As the set $A'_0(q_0) \setminus A_0(q_0)$ is disjoint from the domain E_0 , the point $q(\varepsilon)$ is located outside of E_0 . Point 3 of lemma 8.2 guarantees the existence of a continuous family of marked cycles $\{(\Delta(\varepsilon), q(\varepsilon))\}_{\varepsilon \in \eta_{\max}}$ defined on η_{\max} . For $\varepsilon = \varepsilon_0$ the cycle $(\Delta(\varepsilon_0), q(\varepsilon_0))$ is limit m -fold vertical and has a representative $\delta(\varepsilon_0)$ contained in the domain E_0 . As ε moves on η_{\max} towards 0, it passes through the point ε^* and as a result the point $q(\varepsilon)$ leaves E_0 . Thus, for any $\varepsilon \in \eta_1 \subset \eta_{\max}$ no representative of $(\Delta(\varepsilon), q(\varepsilon))$ is contained in E_0 because all of them pass through the base point $q(\varepsilon)$ which is not in E_0 anymore. The proof of theorem 1 is completed with $\sigma = \eta_{\max}$.

10. CONCLUDING REMARKS

The choice of the family (2) comes into play mostly in the proof of the existence of multi-fold vertical cycles for line fields of type (1). It is specifically designed to facilitate the computation in the first part of the article, where we study the bifurcation of periodic orbits from a resonant parabolic fixed point of the Poincaré map. Establishing the link between a foliation of type (1) and the resonant terms in the normal form of its corresponding Poincaré map seems hard. We can see that in our simple example (2) we have quite involved computations in order to show the non-triviality of the resonant normal form of $P_{a,\varepsilon}$. For the second part of the article,

in which we construct a non-local Poincaré transformation and we study the topological properties and rapid evolution of the multi-fold limit cycles of (2), we do not seem to need that much the explicit form of the foliation of type (1). In fact the central role is played by the polynomial H and its geometric-topological properties. Since H is quite simple, so is its geometry and consequently the topology of the fiber bundle $H : E \rightarrow B$. With some additional modifications one could choose a more complicated polynomial H and carry out similar constructions and prove rapid evolution for more general families of type (1), provided that the existence of a periodic orbit for the Poincaré map is assumed. In [6], or alternatively in [7], the reader could see (1) studied in a more general form. In these works the approach and the general philosophy of the current article are preserved, but the interplay between the topology of the foliation and the dynamics of the Poincaré map are quite more interesting. The map branches and its branching is inherently related to the homotopy class of the loop δ_0 on the surface of a fixed fiber S_{c_0} of H .

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